OASIS seminar 8 March 2024

PARTIAL MARKOV CATEGORIES

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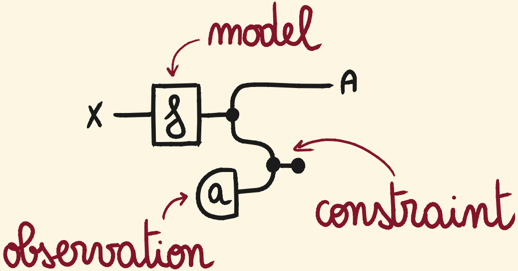
COST EUROPEAN COOPERATION IN SCIENCE & TECHNOLOGY

MOTIVATION

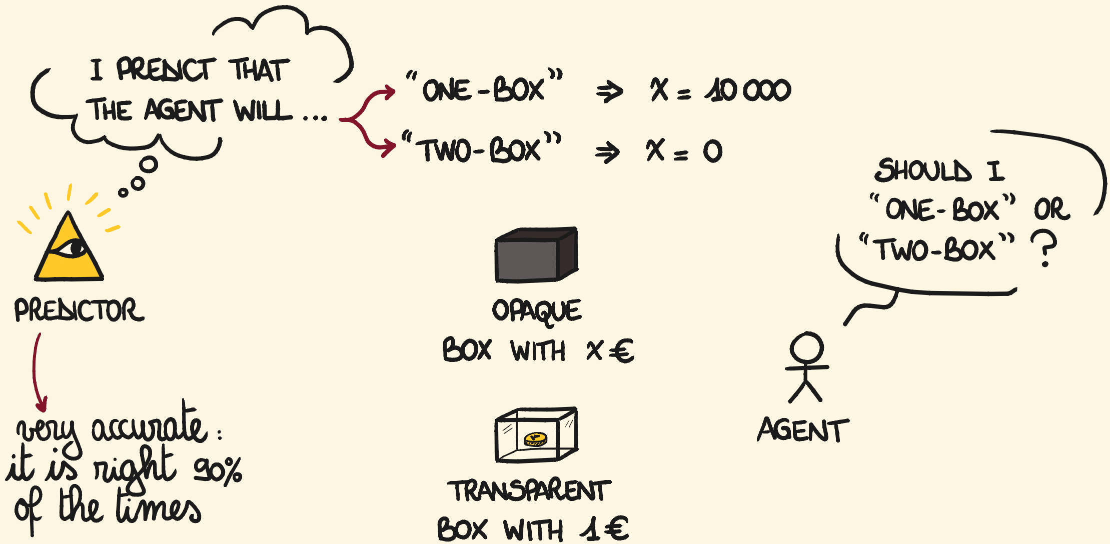
- Find the algebraic structure to express belief updates.
- Markov categories express probabilistic processes.




```
proc(x) = do
  f(x) → a'
  observe_a(a') → ()
  return(a')
```

Updating a model on an observation means restricting the model to scenarios that are compatible with this observation.

NEWCOMB'S PROBLEM



CAUSAL DECISION THEORY

<u>Causal</u> decision theory answers :

"Which action would <u>cause</u> the best-case scenario ?"

Whatever the predictor did,
I get 1€ extra if I two-box
⇒ I will <u>two-box</u>

BEST !

PREDICTOR \ AGENT	ONE-BOX	TWO-BOX
ONE-BOX	10 000 €	10 001€
TWO-BOX	0 €	1 €

EVIDENTIAL DECISION THEORY

Evidential decision theory answers :

"Which action would be evidence for the best-case scenario ?"

My action is evidence for the prediction :
if I one-box I expect 10 000 €,
if I two-box I expect 1 € .
⇒ I will one-box

PREDICTOR \ AGENT	ONE-BOX	TWO-BOX
ONE-BOX	10 000 €	10 001 €
TWO-BOX	0 €	1 €

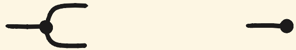
MOST LIKELY

EVIDENTIAL VS CAUSAL DECISION THEORY

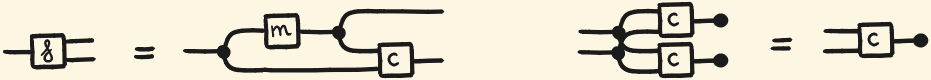
	 CAUSAL DECISION THEORIST	 EVIDENTIAL DECISION THEORIST
EXPECTED UTILITY	0.9×1 € $+ 0.1 \times 10\,001$ € $= 1\,001$ €	$0.9 \times 10\,000$ € $+ 0.1 \times 0$ € $= 9\,000$ €

(DISCRETE) PARTIAL MARKOV CATEGORIES

COPY-DISCARD STRUCTURE



CONDITIONALS



PARTIAL FROBENIUS STRUCTURE

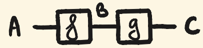
OUTLINE

- Copy-discard categories
- Markov categories
- Cartesian restriction categories
- Partial Markov categories

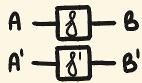
STRING DIAGRAMS

$\mathcal{C}$ symmetric monoidal category

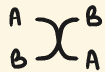
- composition $f;g : A \to C$
  for $f : A \to B$, $g : B \to C$ in $\mathcal{C}$

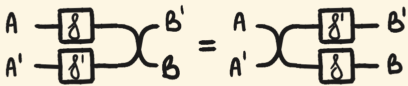


- monoidal product $f \otimes f' : A \otimes A' \longrightarrow B \otimes B'$
  for $f : A \to B$, $f' : A' \to B'$ in $\mathcal{C}$



- symmetry $\sigma_{A,B} : A \otimes B \longrightarrow B \otimes A$





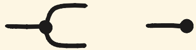
(naturality)

EXAMPLES

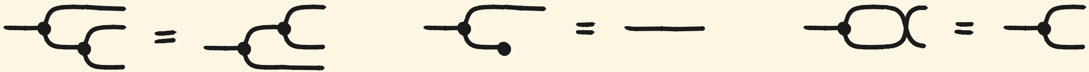
- $(\mathbf{Set}, \times, \{*\})$: sets and functions
  $f : A \to B$ is a function
- $(\mathbf{Par}, \times, \{*\})$: sets and partial functions
  $f : A \nrightarrow B$ is a function $f : A \to B+1$
- $(\mathcal{K}\ell(\mathcal{D}), \times, \{*\})$: sets and stochastic functions
  $f : A \nrightarrow B$ is a function $f : A \to \mathcal{D}(B)$
- $(\mathbf{Rel}, \times, \{*\})$: sets and relations
  $f : A \nrightarrow B$ is a function $f : A \to \mathcal{P}(B)$
- $(\mathcal{K}\ell(\mathcal{D}_{\leq 1}), \times, \{*\})$: sets and partial stochastic functions
  $f : A \nrightarrow B$ is a function $f : A \to \mathcal{D}(B+1)$

COPY-DISCARD CATEGORIES

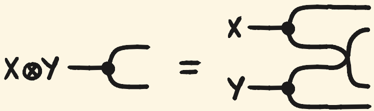
A copy-discard category is a symmetric monoidal category where every object is a uniform cocommutative comonoid.



COCOMMUTATIVE COMONOID



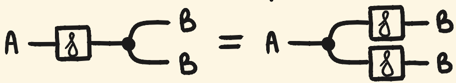
UNIFORMITY



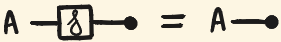
[Corradini & Gadducci 1999]

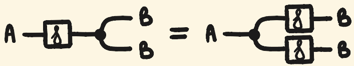
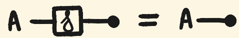
DETERMINISTIC & TOTAL MAPS

Deterministic maps can be copied.



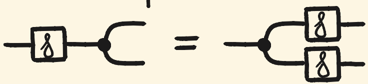
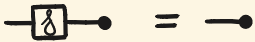
Total maps can be discarded.



EXAMPLES		
$(\mathcal{S}et, \times, \{*\})$	✓	✓
$(\mathcal{P}ar, \times, \{*\})$	✓	✗
$(\mathcal{K}\ell(\mathcal{D}), \times, \{*\})$	✗	✓
$(\mathcal{R}el, \times, \{*\})$	✗	✗
$(\mathcal{K}\ell(\mathcal{D}_{\leq 1}), \times, \{*\})$	✗	✗

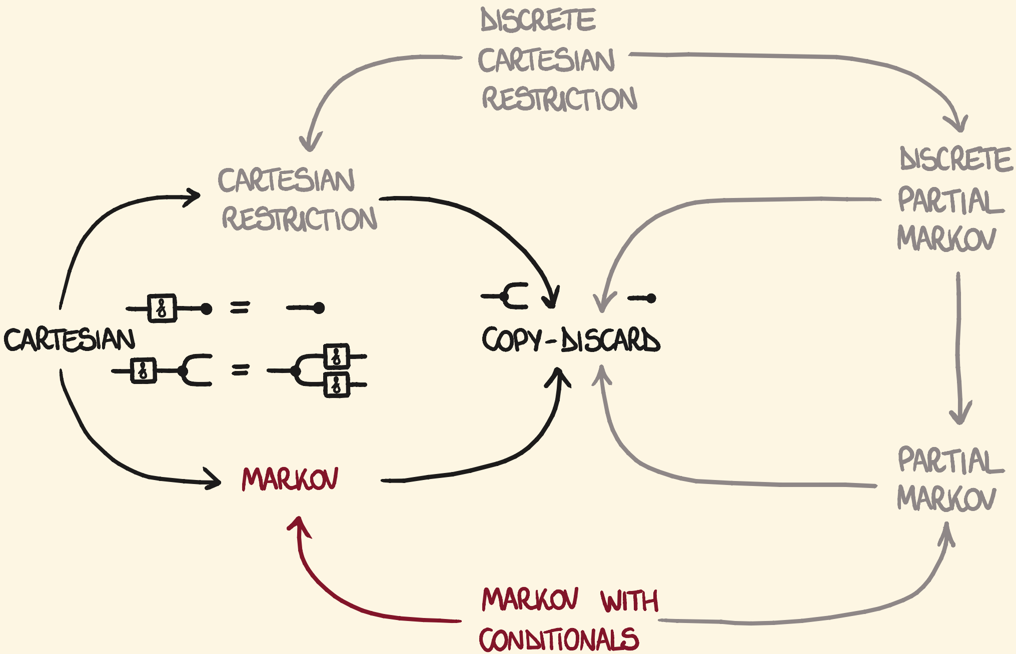
FOX'S THEOREM

A copy-discard category is cartesian if and only if all morphisms are deterministic and total,

 and  for all f.

[Fox 1976]

OUTLINE



DISCRETE CARTESIAN RESTRICTION
CARTESIAN RESTRICTION
DISCRETE PARTIAL MARKOV
CARTESIAN
COPY-DISCARD
MARKOV
PARTIAL MARKOV
MARKOV WITH CONDITIONALS
=
=

PROBABILISTIC PROCESSES

Markov categories express probabilistic processes, for example

- throwing a coin

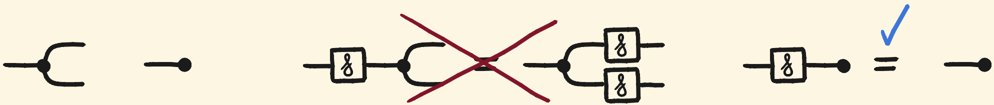
- tomorrow's weather given today's clouds

- developing cancer given smoking habits

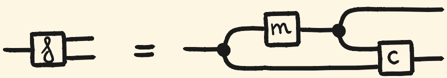
MARKOV CATEGORIES & CONDITIONALS

A Markov category with conditionals is a copy-discard category with conditionals where all morphisms are total.

COPY - DISCARD STRUCTURE



CONDITIONALS



[Fritz 2020]

FINITARY DISTRIBUTIONS

A *finitary distribution* $\sigma \in \mathcal{D}(A)$ is a function $\sigma : A \to [0,1]$ such that

- its support, $\mathrm{supp}(\sigma) := \{a \in A \mid \sigma(a) > 0\}$, is finite, and
- its total probability mass is 1, $\sum_{a \in A} \sigma(a) = 1$.

A morphism $X - \boxed{f} - A$ in $Kl\mathcal{D}$ is a function $X \to \mathcal{D}(A)$

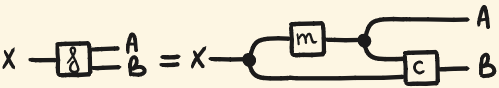
$$f(a \mid x) = \text{"probability of } a \text{ given } x\text{"}$$

composition is

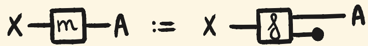
$$X - \boxed{f} - \boxed{g} - B \ (b \mid x) := \sum_{a \in A} f(a \mid x) \cdot g(b \mid a)$$

CONDITIONALS

Kl$\mathcal{D}$ has conditionals.



$$m(a|x) := \sum_{b\in B} f(a,b|x)$$



$$c(b|a,x) := \begin{cases} \dfrac{f(a,b|x)}{m(a|x)} & \text{if } m(a|x) \neq 0 \\ \sigma(b) & \text{if } m(a|x) = 0 \end{cases}$$

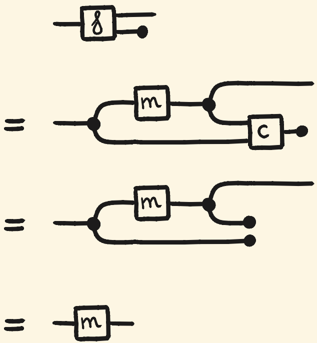
any distribution on B

⇝ conditionals are not unique and they cannot be

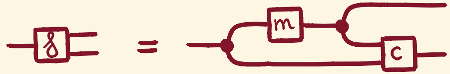
MARGINALS IN MARKOV CATEGORIES

Marginals in Markov categories are as expected :

PROOF



conditionals :



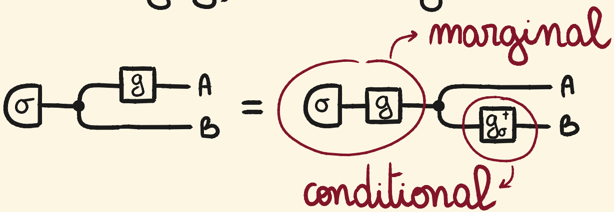
totality

BAYES INVERSION

The Bayes inversion of a channel $g : B \to A$ with respect to a distribution $\sigma : I \to B$ is classically defined as

$$g^{\dagger}_{\sigma}(b|a) := \frac{g(a|b)\,\sigma(b)}{\sum_{b' \in B} g(a|b')\,\sigma(b')}$$

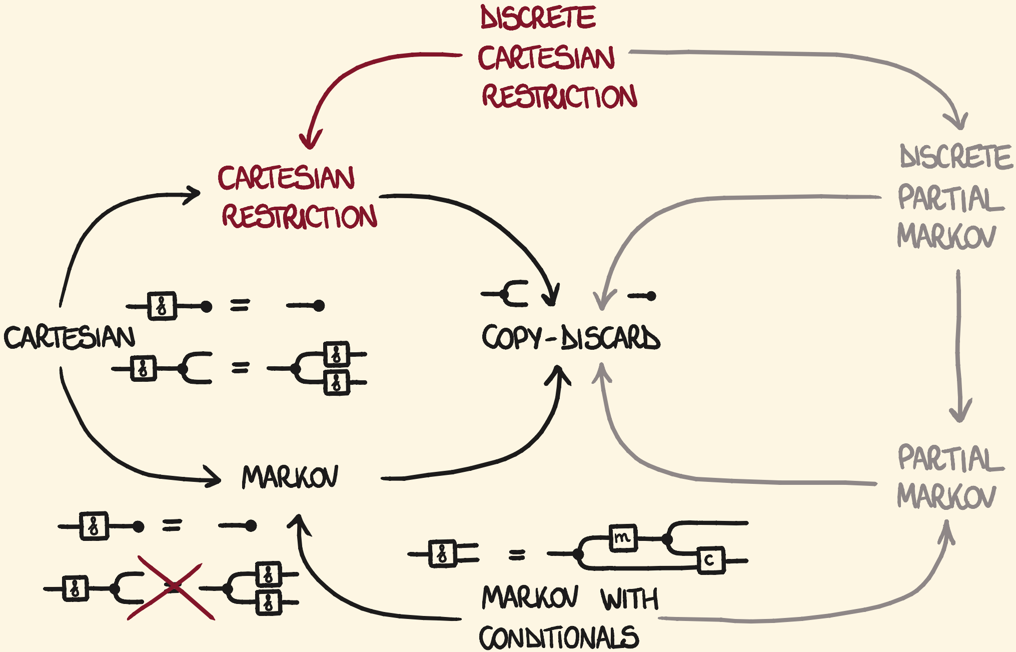
In a Markov category, it is a $g^{\dagger}_{\sigma} : A \to B$ such that



Bayes inversions are instances of conditionals.

[Cho & Jacobs 2019]

OUTLINE



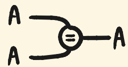
DISCRETE CARTESIAN RESTRICTION
CARTESIAN RESTRICTION
DISCRETE PARTIAL MARKOV
CARTESIAN
COPY-DISCARD
MARKOV
PARTIAL MARKOV
m
c
MARKOV WITH CONDITIONALS

PARTIAL PROCESSES

Cartesian restriction categories express partial computations, for example

- computing $\frac{1}{x}$

- checking equality

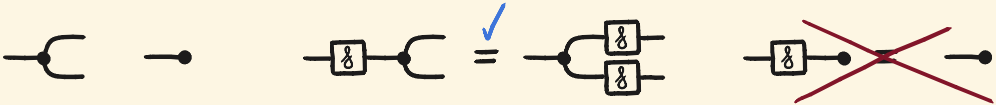


- non-terminating computations

CARTESIAN RESTRICTION CATEGORIES

A cartesian restriction category is a copy-discard category where all morphisms are deterministic.

COPY - DISCARD STRUCTURE



[Cockett & Lack 2003, 2007]

PARTIAL FUNCTIONS

$(\mathsf{Par}, \times, \{*\})$ is a cartesian restriction category

- objects are sets $A, B, C, \ldots$
- morphisms are partial functions $f : A \rightharpoonup B$, $g : B \rightharpoonup C, \ldots$
  i.e. functions $f : A \to B+1$, $g : B \to C+1, \ldots$
- composition is

$$f;g\,(a) := \begin{cases} g(f(a)) & \text{if } f(a) \neq \perp \\ \perp & \text{if } f(a) = \perp \end{cases}$$

- monoidal product is

$$f \times f'\,(a,a') := \begin{cases} (f(a), f'(a')) & \text{if } f(a) \neq \perp \text{ and } f'(a') \neq \perp \\ \perp & \text{if } f(a) = \perp \text{ or } f'(a') = \perp \end{cases}$$

PREDICATES, DOMAINS, RESTRICTIONS

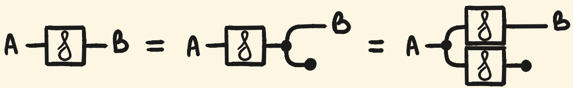
Morphisms $q : A \to 1$ in Par are predicates.

$(a) = \begin{cases} * & \text{if } a \text{ satisfies } q \\ \perp & \text{if } a \text{ does not satisfy } q \end{cases}$

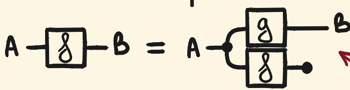
The domain of 

is the predicate 

.



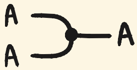
The restriction preorder on morphisms is

$f \leq g$ iff 

g restricted to the domain of f

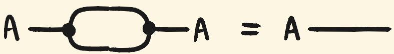
EQUALITY CHECK

Par has equality checks.

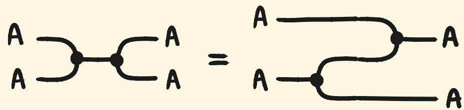


$$(a, a') := \begin{cases} a & \text{if } a = a' \\ \perp & \text{if } a \neq a' \end{cases}$$

Equality checks interact with the comonoid structure.



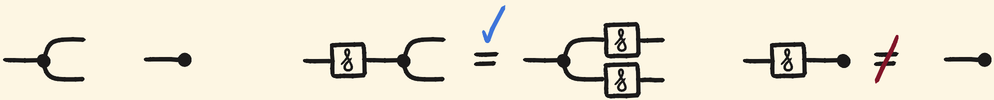
and



CONSTRAINTS VIA PARTIAL FROBENIUS

A discrete cartesian restriction category is a copy-discard category with comparators where all morphisms are deterministic.

COPY-DISCARD STRUCTURE

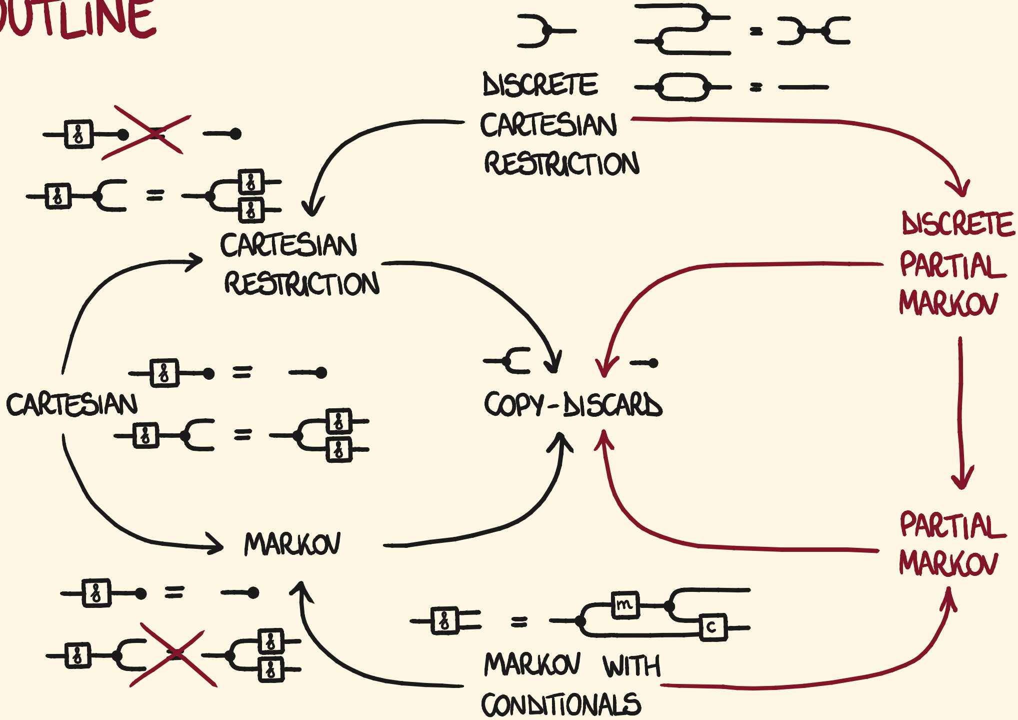


PARTIAL FROBENIUS STRUCTURE

COMPARATOR

[Cockett, Guo & Hofstra 2012, Di Liberti, Loregian, Nester & Sobociński 2020]

OUTLINE



DISCRETE CARTESIAN RESTRICTION
CARTESIAN RESTRICTION
DISCRETE PARTIAL MARKOV
CARTESIAN
COPY-DISCARD
PARTIAL MARKOV
MARKOV
MARKOV WITH CONDITIONALS
m
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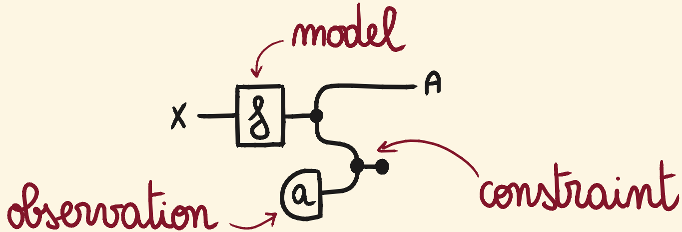
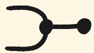
PARTIALITY FOR OBSERVATIONS

Updating a model on an observation means restricting the model to scenarios that are compatible with this observation.



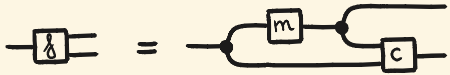
constraints  cannot be total computations
because  ≠  .

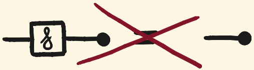
OVERVIEW

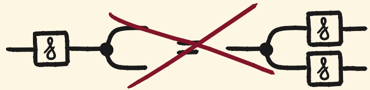
Combine Markov and cartesian restriction categories to express partial stochastic processes.

cartesian restriction

Markov with conditionals

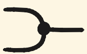


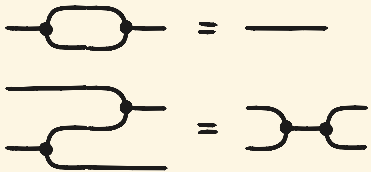




Add the discrete structure to express equality checking.

discrete cartesian restriction



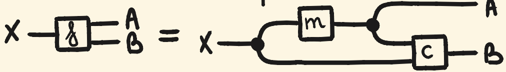


OUTLINE

- (discrete) partial Markov categories
- updating
- normalising

DROPPING TOTALITY

We want to keep the nice marginals of Markov categories.



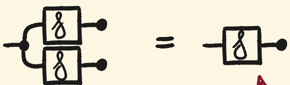
Should we ask conditionals to be total? ✗ NO

⇝ too strong: total conditionals fail to exist in $Kl(\mathcal{D}_{\leq 1})$.

Can we ask conditionals to be quasi-total? ✓ YES

⇝ sweet spot: quasi-total conditionals usually exist and give nice marginals.

QUASI-TOTAL MORPHISM (in a copy-discard category)



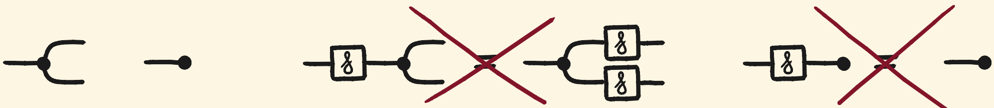
⇝ failure is deterministic

domain of definition

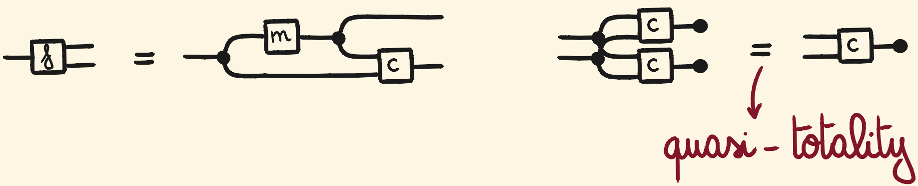
PARTIAL MARKOV CATEGORIES

A partial Markov category is a copy-discard category with quasi-total conditionals.

COPY-DISCARD STRUCTURE



CONDITIONALS



[EDL & Román, 2023]

SUBDISTRIBUTIONS

A <u>subdistribution</u> σ on A is a distribution on $A+1$:

$\sigma \in \mathcal{D}_{\leq 1}(A)$ is a function $\sigma : A \to [0,1]$ such that

- its support, $\mathrm{supp}(\sigma) := \{a \in A \mid \sigma(a) > 0\}$, is finite, and
- its total probability mass is at most 1, $\sum_{a \in A} \sigma(a) \leq 1$.

A morphism $X - \boxed{f} - A$ in $\mathrm{Kl}\mathcal{D}_{\leq 1}$ is a function $X \to \mathcal{D}_{\leq 1}(A)$

$f(a|x)$ = "probability of a given x"

$f(\perp|x)$ = "probability of failure"

Composition is

$$X - \boxed{f} - \boxed{g} - B \;(b|x) := \sum_{a \in A} f(a|x) \cdot g(b|a)$$

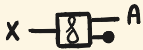
$$X - \boxed{f} - \boxed{g} - B \;(\perp|x) := \sum_{a \in A} f(a|x) \cdot g(\perp|a) + f(\perp|x)$$

CONDITIONALS IN SUBDISTRIBUTIONS

A quasi-total morphism $g : X \to B$ is a function $g : X \to \mathcal{D}B + 1$.

The marginal of $f : X \to A \otimes B$ is

$(a|x) =$

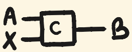


$$(a|x) = \sum_{b \in B} f(a, b|x)$$

$(\perp|x) =$

$$(\perp|x) = f(\perp|x)$$

A conditional of f is:



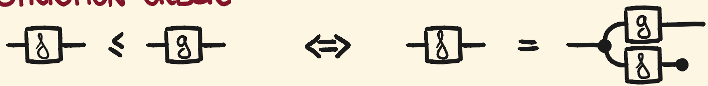
$$(b|a,x) = \begin{cases} \dfrac{f(a,b|x)}{m(a|x)} & m(a|x) \neq 0 \\ 0 & m(a|x) = 0 \end{cases}$$

$$(\perp|a,x) = \begin{cases} 0 & m(a|x) \neq 0 \\ 1 & m(a|x) = 0 \end{cases}$$

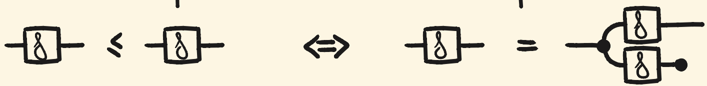
MINIMAL CONDITIONALS

$\mathcal{C}$ partial Markov category

RESTRICTION ORDER



→ this is a partial order on quasi-total morphisms



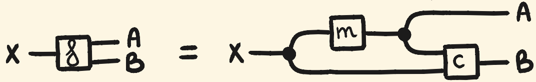
(reflexivity) (quasi-totality)

MINIMAL CONDITIONALS

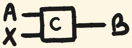
$\mathcal{C}$ has minimal conditionals if, for every , the partial order on its quasi-total conditionals has a minimal element.

MINIMAL CONDITIONALS IN SUBDISTRIBUTIONS

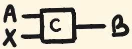
A quasi-total morphism $g : X \to B$ is a function $g : X \to \mathcal{D}B + 1$.



The conditionals of f are:



$$(b \mid a, x) = \begin{cases} \dfrac{f(a, b \mid x)}{m(a \mid x)} & m(a \mid x) \neq 0 \\ \sigma(b) & m(a \mid x) = 0 \end{cases}$$



$$(\perp \mid a, x) = \begin{cases} 0 & m(a \mid x) \neq 0 \\ \sigma(\perp) & m(a \mid x) = 0 \end{cases}$$

for some $\sigma \in \mathcal{D}(B) + 1$.

$\Rightarrow$ The minimal choice for σ is $\begin{cases} \sigma(b) = 0 & \text{for } b \in B \\ \sigma(\perp) = 1 \end{cases}$.

$\rightsquigarrow$ fail on unexpected observations

EXAMPLES : PARTIAL STOCHASTIC PROCESSES

A <u>partial stochastic process</u> is a stochastic process that may fail.

a Markov category with conditionals

↳ Maybe monad

Partial stochastic processes form a partial Markov category.

PROPOSITION

$\begin{cases} \mathcal{C} \text{ Markov category with conditionals and coproducts} \\ \text{Some ugly technical conditions} \end{cases}$

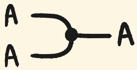
$\Rightarrow \mathrm{Kl}(\cdot + 1)$ is a partial Markov category.

EXAMPLES

- $\mathrm{Kl}(\mathcal{D}(\cdot + 1))$ ⇝ finitary subdistributions
- $\mathrm{Kl}(\mathcal{G}iry_B(\cdot + 1))$ ⇝ subdistributions on standard Borel spaces

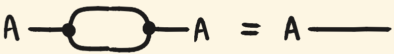
EQUALITY CHECK

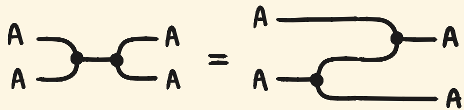
$\mathrm{Kl}\,\mathcal{D}_{\leq 1}$ has equality checks.



$(a, a') := \begin{cases} \delta_a & \text{if } a = a' \\ \delta_\perp & \text{if } a \neq a' \end{cases}$

Equality checks interact with the comonoid structure.

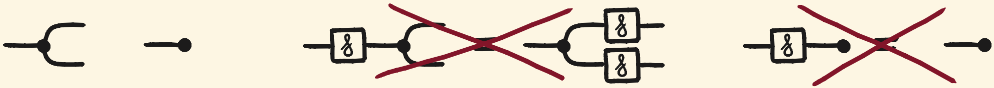


and


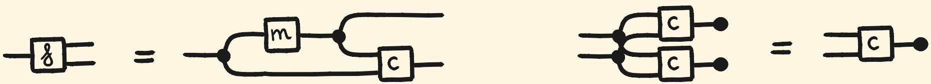
DISCRETE PARTIAL MARKOV CATEGORIES

A discrete partial Markov category is a copy-discard category with quasi-total conditionals and comparators.

COPY - DISCARD STRUCTURE



CONDITIONALS



PARTIAL FROBENIUS STRUCTURE

COMPARATOR

[EDL & Román, 2023]

OUTLINE

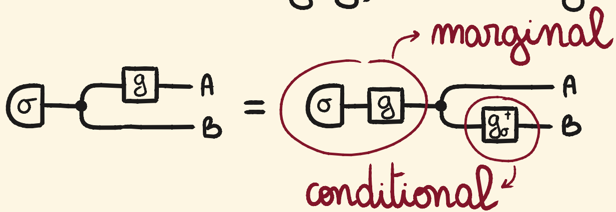
- (discrete) partial Markov categories
- [updating]
- normalising

BAYES INVERSION

The Bayes inversion of a channel $g : B \to A$ with respect to a distribution $\sigma : I \to B$ is classically defined as

$$g^\dagger_\sigma(b|a) := \frac{g(a|b)\,\sigma(b)}{\sum_{b' \in B} g(a|b')\,\sigma(b')}$$

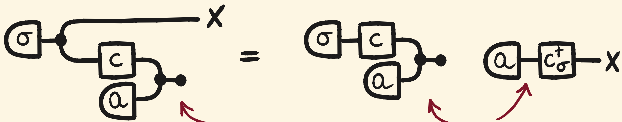
In a partial Markov category, it is a $g^\dagger_\sigma : A \to B$ such that



Bayes inversions are instances of quasi-total conditionals.

SYNTHETIC BAYES THEOREM

A deterministic observation $a : I \to A$ from a prior $\sigma : I \to X$ through a channel $c : X \to A$ determines an update proportional to the Bayes inversion $c^{\dagger}_{\sigma}$ evaluated on a.

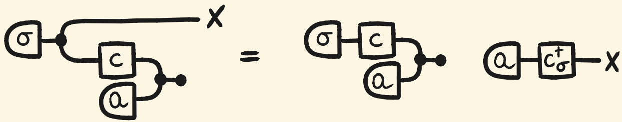


$$P(X{=}x \mid A{=}a) = \frac{P(A{=}a \mid X{=}x) \cdot P(X{=}x)}{\sum_{y \in X} P(A{=}a \mid X{=}y) \cdot P(X{=}y)}$$

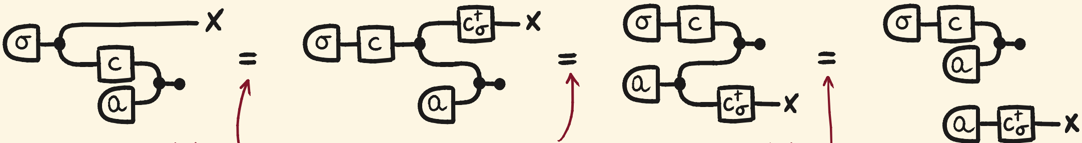
classical formula for Bayes theorem

SYNTHETIC BAYES THEOREM

A deterministic observation $a : I \to A$ from a prior $\sigma : I \to X$ through a channel $c : X \to A$ determines an update proportional to the Bayes inversion $c^\dagger_\sigma$ evaluated on a.



PROOF



conditionals Frobenius determinism

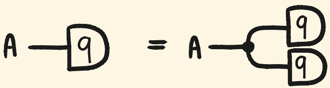
□

PREDICATES & DOMAINS

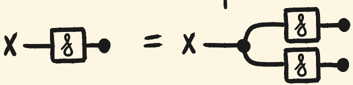
Morphisms $q : A \to 1$ in $\mathrm{Kl}(\mathcal{D}(\bullet + 1))$ are 'fuzzy' predicates.

 $(* \mid a)$ ⇝ probability of a being true

Deterministic predicates are classical predicates.

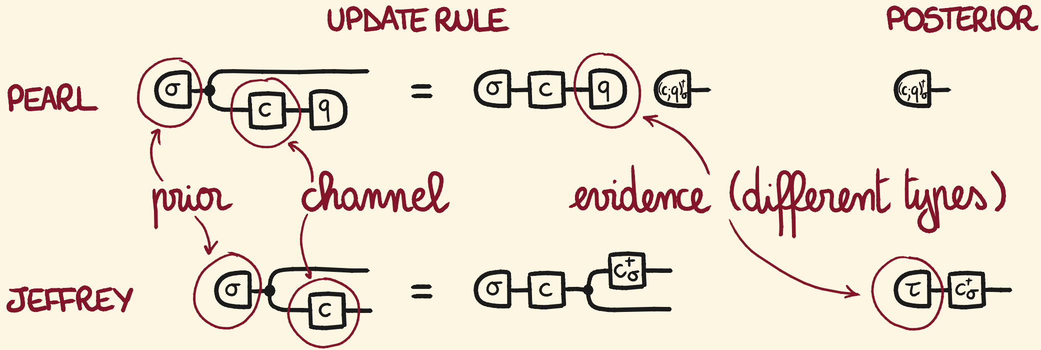
 $\Rightarrow$ q is a classical predicate

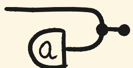
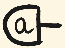
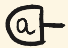
Quasi-total morphisms have a domain.

 ⇝ domain of f

probability of failure of f

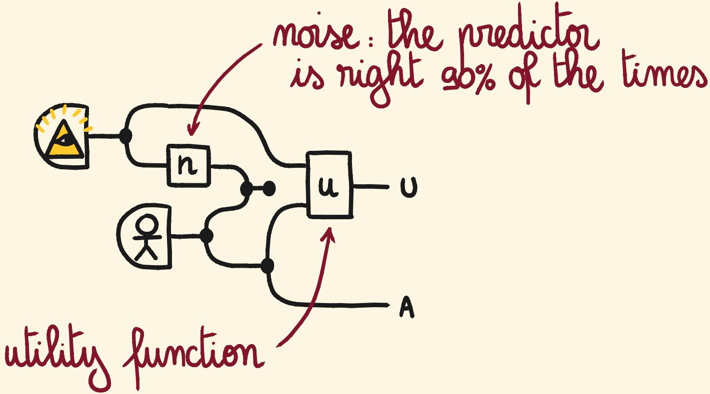
PEARL'S VS JEFFREY'S UPDATES



Pearl's update on  coincides with Jeffrey's update on , whenever  is deterministic.

[Jacobs 2019]

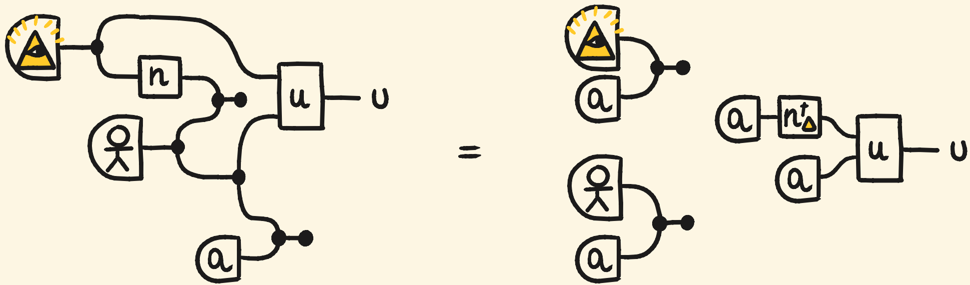
NEWCOMB'S PROBLEM CATEGORICALLY



AGENT / PREDICTOR	ONE-BOX	TWO-BOX
ONE-BOX	10 000 €	10 001 €
TWO-BOX	0 €	1 €

SOLVING NEWCOMB'S PROBLEM

Evidential decision theory asks:
"Which action would be evidence for the best-case scenario?"
i.e. "Which action maximises the average of the state below?"



OUTLINE

- (discrete) partial Markov categories
- updating

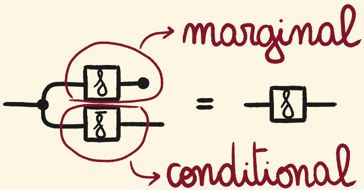
[- normalising]

NORMALISATION

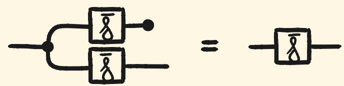
The normalisation of a partial channel $f : X \to A$ is classically defined as

$$\bar{f}(a|x) := \frac{f(x|a)}{1 - f(\perp|a)}$$

In a partial Markov category, it is a $\bar{f} : X \to A$ such that



and



Normalisations are instances of quasi-total conditionals.

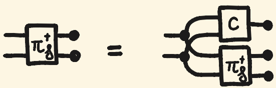
QUASI-TOTAL CONDITIONALS ARE NECESSARY

THEOREM

A copy-discard category has quasi-total conditionals iff it has bayesian inversions and normalisations.

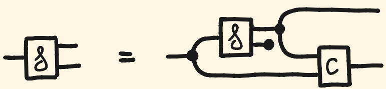
PROOF IDEA

bayesian inversion of



normalisation of $\pi^\dagger_g$

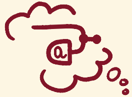
$\Rightarrow$



$\square$

PROCESSES WITH EXACT OBSERVATIONS

We can add exact observations to any Markov category:

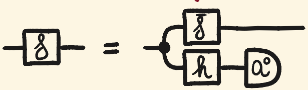


$\mathrm{exOb}(\mathcal{C}) = \left(\mathcal{C} + \{ A - a^{\circ} \mid a - A \text{ deterministic} \}\right) /$ 

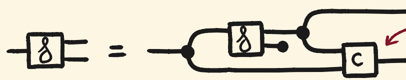
embeds faithfully into $\left(\mathcal{C} +\right.$  $\left.\right)/$ partial Frobenius

Conditionals and normalisations are computed in $\mathcal{C}$

normalisation of f

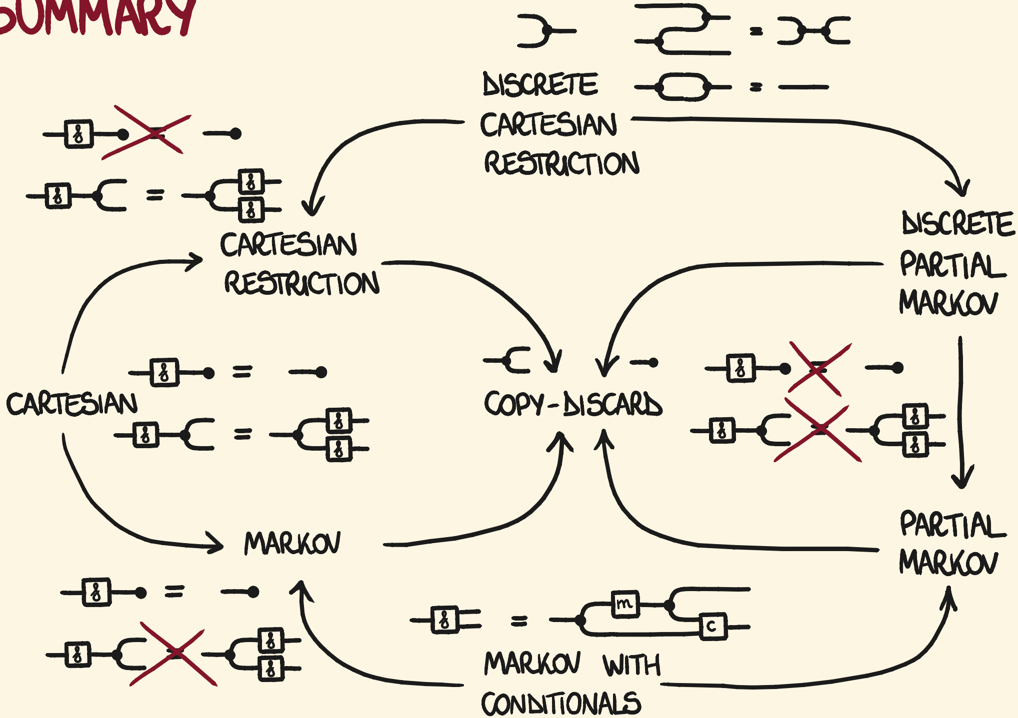


conditional of f



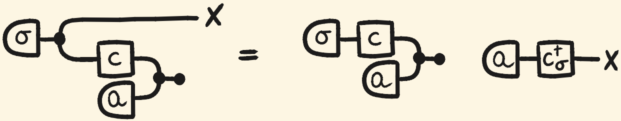
$\Rightarrow$ $\mathrm{exOb}(\mathcal{C})$ is a partial Markov category.

SUMMARY


DISCRETE CARTESIAN RESTRICTION
CARTESIAN RESTRICTION
DISCRETE PARTIAL MARKOV
CARTESIAN
COPY-DISCARD
MARKOV
PARTIAL MARKOV
MARKOV WITH CONDITIONALS
m
c

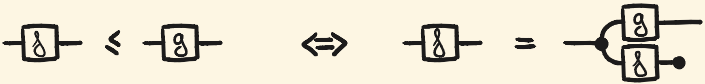
CONCLUSIONS & FUTURE WORK

- Partial Markov categories are for updating on observations.



Synthetic Bayes theorem

- Is there a bicategorical structure hiding?



Restriction preorder