

Logicon

27 May 2026

MONOIDAL CATEGORIES FOR PROBABILISTIC REASONING

Elena Di Larore

Tallinn University of Technology

OVERVIEW

- categorical structures for program semantics
 - ↳ monoidal and copy-discard categories
 - ⇒ background material
- Focus on probabilistic programs and synthetic probability
 - ↳ partial Markov categories and synthetic Bayes' theorem
 - ⇒ joint work with Mario Román, Márk Széles
Paweł Sobociński, Filippo Bonchi, Sam Staton

CATEGORICAL STRUCTURES AS MODELLING LANGUAGES

Von Neumann's trick

unbiased (p)

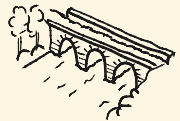
```
x := 0
y := 0
while x = y
  x ← coin( $p$ )
  y ← coin( $p$ )
return x
```

Euclidean algorithm

gcd(m, n)

```
while  $n \neq 0$ 
  t := n
  n := m mod n
  m := t
return m
```

How do we ensure that programs do what they are meant to?
→ write mathematical models of their behaviour
and then formally reason about the models



CATEGORICAL STRUCTURES AS MODELLING LANGUAGES

- categorical semantics of programming languages
 - ↳ abstraction and compositionality for conceptual understanding
- Programs with (commutative) effects
 - ↳ quantum programs
 - nondeterministic programs
 - probabilistic programs

What are good categorical structures to reason about them?

MORPHISMS AS PROGRAMS

$$f: A \rightarrow B$$



$$g: B \rightarrow C$$



$$f': A' \rightarrow B'$$



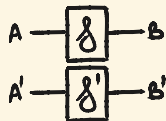
- Sequential composition.

$$f; g: A \rightarrow C$$



- Parallel composition.

$$f \otimes f': A \otimes A' \rightarrow B \otimes B'$$



- Multiple/no inputs/outputs.

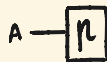
$$h: A_1 \otimes A_2 \rightarrow B_1 \otimes B_2 \otimes B_3$$



$$\Delta: I \rightarrow A \otimes B$$



$$\eta: A \rightarrow I$$



MONOIDAL CATEGORIES

A strict monoidal category, $(\mathcal{C}, \otimes, I)$, is a category, $\mathcal{C}$, with two functors, $(-\otimes-): \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ and $I: 1 \rightarrow \mathcal{C}$, satisfying associativity and unitality:

$$\begin{cases} (A \otimes B) \otimes C = A \otimes (B \otimes C) \\ A \otimes I = A = I \otimes A \end{cases}$$

$$\begin{cases} (f \otimes g) \otimes h = f \otimes (g \otimes h) \\ f \otimes \text{id}_I = f = \text{id}_I \otimes f \end{cases}$$

$\leadsto$ we can omit parentheses: $f \otimes g \otimes h$

MONOIDAL CATEGORIES

A monoidal category, $(\mathcal{C}, \otimes, I, \alpha, \lambda, e)$, is a category, $\mathcal{C}$, with two functors, $(-\otimes-): \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ and $I: 1 \rightarrow \mathcal{C}$, satisfying associativity and unitality "up to isomorphism":

there are natural isomorphisms

$$\begin{cases} \alpha_{A,B,C}: (A \otimes B) \otimes C \rightarrow A \otimes (B \otimes C) \\ \lambda_A: I \otimes A \rightarrow A \\ e_A: A \otimes I \rightarrow A \end{cases}$$

that are coherent



there is a unique coherence isomorphism between any two parenthatisations of the same list of objects, e.g. $A \otimes ((B \otimes C) \otimes D) \cong (A \otimes B) \otimes (C \otimes D)$

$$\begin{aligned} & \circ \circ \left\{ \begin{aligned} (A \otimes B) \otimes C &\cong A \otimes (B \otimes C) \\ A \otimes I &\cong A \cong I \otimes A \end{aligned} \right. \\ & \left\{ \begin{aligned} (A \otimes B) \otimes C &\xrightarrow{\alpha} A \otimes (B \otimes C) \\ (f \otimes g) \otimes h \downarrow &= \downarrow f \otimes (g \otimes h) \\ (A' \otimes B) \otimes C' &\xrightarrow{\alpha} A' \otimes (B \otimes C') \end{aligned} \right. \end{aligned}$$

$$\begin{array}{ccccc} A \otimes I & \xrightarrow{e} & A & \xleftarrow{\lambda} & I \otimes A \\ f \otimes \text{id} \downarrow & = & f \downarrow & = & \downarrow \text{id} \otimes g \\ A' \otimes I & \xrightarrow{e} & A' & \xleftarrow{\lambda} & I \otimes A' \end{array}$$

MONOIDAL CATEGORIES

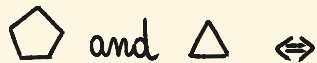
Pentagon  and triangle  equations

$$\begin{array}{ccc}
 & (A \otimes (B \otimes C)) \otimes D & \\
 \alpha_{A,B,C} \otimes \text{id} \nearrow & & \searrow \alpha_{A,B \otimes C,D} \\
 ((A \otimes B) \otimes C) \otimes D & \equiv & A \otimes ((B \otimes C) \otimes D) \\
 \alpha_{A \otimes B,C,D} \searrow & & \swarrow \text{id} \otimes \alpha_{B,C,D} \\
 (A \otimes B) \otimes (C \otimes D) & \xrightarrow{\alpha_{A,B,C \otimes D}} & A \otimes (B \otimes (C \otimes D))
 \end{array}$$

$$\begin{array}{ccc}
 & A \otimes B & \\
 e_A \otimes \text{id} \nearrow & & \nwarrow \text{id} \otimes e_B \\
 (A \otimes I) \otimes B & \xrightarrow{\alpha_{A,I,B}} & A \otimes (I \otimes B)
 \end{array}$$

COHERENCE AND STRICTIFICATION

THEOREM (coherence)



There is a unique coherence isomorphism between any two parenthisations of the same list of objects

THEOREM (strictification)

Every monoidal category is monoidally equivalent to a strict one.

CONSEQUENCE : we can forget about parentheses.

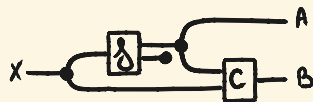
[MacLane (1971)]

STRING DIAGRAMS

Intuitive and formal syntax for monoidal categories

$$\begin{array}{c}
 A \text{ --- } \boxed{f} \text{ --- } B \\
 A' \text{ --- } \boxed{g} \text{ --- } B'
 \end{array}
 =
 \begin{array}{c}
 A \text{ --- } \boxed{f} \text{ --- } B \\
 A' \text{ --- } \boxed{g} \text{ --- } B'
 \end{array}
 =
 \begin{array}{c}
 A \text{ --- } \boxed{f} \text{ --- } B \\
 A' \text{ --- } \boxed{g} \text{ --- } B'
 \end{array}$$

$$(f \otimes \text{id}) ; (\text{id} \otimes g) = f \otimes g = (\text{id} \otimes g) ; (f \otimes \text{id})$$



$$\nu_x ; ((f ; (\text{id}_A \otimes \varepsilon_B) ; \nu_A) \otimes \text{id}_x) ; (\text{id}_A \otimes c)$$

[Joyal and Street (1991)]

EXAMPLES

- Set and Par

objects = sets

morphisms = functions / partial functions

- Rel

objects = sets

morphisms $A \rightarrow B$ = relations $R \subseteq A \times B$

- FHilb

objects = finite dimensional Hilbert spaces

morphisms = linear maps

- Stoch and $\text{Stoch}_{\leq}$

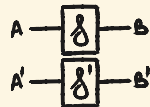
objects = sets / measurable spaces

morphisms = finitely or countably supported / measurable
(sub)stochastic kernels

DISCRETE (SUB)STOCHASTIC KERNELS

- objects = sets, morphisms $A \rightarrow B$ = functions $A \rightarrow \mathcal{D}(B)$
- $\mathcal{D}(B) :=$ "set of finitely-supported subdistributions on B "
$$:= \{ \sigma : B \rightarrow [0, 1] \mid \text{supp}(\sigma) \text{ finite} \wedge \sum_{b \in B} \sigma(b) \leq 1 \}$$

$$\leadsto \text{supp}(\sigma) := \{ b \in B \mid \sigma(b) > 0 \}$$
- $f(b|a) := f(a)(b) \in [0, 1]$
 $\leadsto$ "probability that f outputs b given input a "
- $f: A \rightarrow B$, $g: B \rightarrow C$, $f': A' \rightarrow B'$
 $f; g(c|a) := \sum_{b \in B} f(b|a) \cdot g(c|b)$
 $f \otimes f'(b, b'|a, a') := f(b|a) \cdot f'(b'|a')$



STRUCTURE OF STOCHASTIC KERNELS

- states and predicates

$$\delta : I \rightarrow A \otimes B \quad \boxed{\delta} \vdash^A B$$

$\leadsto$ joint distribution

$$\mu : A \rightarrow I \quad A \vdash \boxed{\mu}$$

$\leadsto$ "fuzzy" predicate

- marginal distributions by discarding

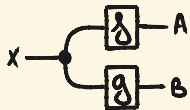
$$\boxed{\delta} \vdash^A \bullet$$

$$\leadsto P(A) = \sum_{b \in B} P(A, B=b)$$

$$x \vdash \boxed{\mu} \vdash^A \bullet$$

$$\leadsto P(A|X) = \sum_{b \in B} P(A, B=b | X)$$

- common causes by copying

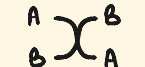


$$\leadsto A \perp B | X$$

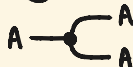
COPY-DISCARD CATEGORIES

A copy-discard category is a symmetric monoidal category where every object has a coherent cocommutative comonoid structure.

symmetries


$$A \otimes B \rightarrow B \otimes A$$

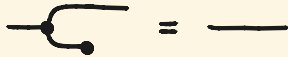
copy


$$A \rightarrow A \otimes A$$

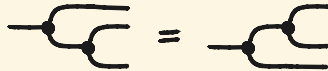
discard


$$A \rightarrow \mathbb{I}$$

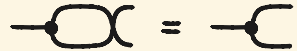
counitality


$$= \text{strand}$$

coassociativity


$$= \text{strand}$$

cocommutativity


$$= \text{strand}$$

DETERMINISTIC AND TOTAL MORPHISMS

- Deterministic morphisms can be copied.

$$A \text{ --- } \boxed{\delta} \text{ --- } \begin{matrix} B \\ B \end{matrix} = A \text{ --- } \begin{matrix} \boxed{\delta} \text{ --- } B \\ \boxed{\delta} \text{ --- } B \end{matrix}$$

- Total morphisms can be discarded.

$$A \text{ --- } \boxed{\delta} \text{ --- } \bullet = A \text{ --- } \bullet$$

EXAMPLES

$$A \text{ --- } \boxed{\delta} \text{ --- } \begin{matrix} B \\ B \end{matrix} = A \text{ --- } \begin{matrix} \boxed{\delta} \text{ --- } B \\ \boxed{\delta} \text{ --- } B \end{matrix}$$

$$A \text{ --- } \boxed{\delta} \text{ --- } \bullet = A \text{ --- } \bullet$$

Set

✓

✓

Par

✓

✗

$\begin{matrix} \text{---} \bullet \\ \text{---} \bullet \end{matrix} \neq \text{---} \bullet$

Stoch

✗ $\begin{matrix} \text{---} \bullet \\ \text{---} \bullet \end{matrix} \neq \begin{matrix} \boxed{\delta} \text{ --- } \bullet \\ \boxed{\delta} \text{ --- } \bullet \end{matrix}$

✓

Rel

✗

✗

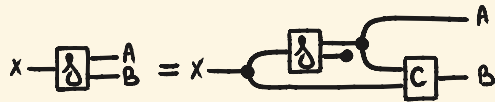
Stoch_≤

✗

✗

CONDITIONALS

A morphism $f: X \rightarrow A \otimes B$ in a copy-discard category has a conditional if there is a morphism $c: A \otimes X \rightarrow B$ decomposing f as below.



• In $\text{Stoch}_{\leq}$, $f(a, b | x) = \sum_{b' \in B} f(a, b' | x) \cdot c(b | a, x)$

$$\Rightarrow c(b | a, x) = \frac{f(a, b | x)}{\sum_{b' \in B} f(a, b' | x)} \quad \text{when } \sum_{b' \in B} f(a, b' | x) > 0$$

and not uniquely defined otherwise

[Dritz (2020); Cho, Jacobs (2019)]

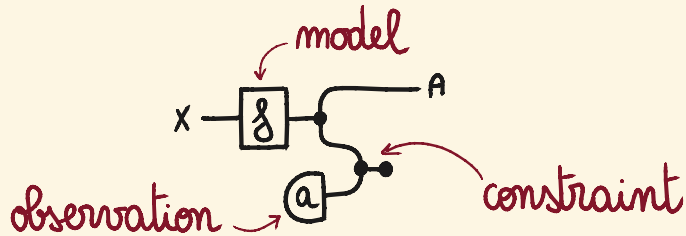
MARKOV CATEGORIES AND ADJACENT STRUCTURES

- Markov categories and conditionals
[Choi, Jacobs (2019); Fritz (2020)]
- Theorems from probability theory
[Fritz, Gonda, Perrone (2021); Fritz, Rischel (2020); Perrone, Van Belle (2024)]
- Probabilistic programming constructions
[Stein, Staton (2023); Moss, Perrone (2022); Heunen, Kammara, Staton, Yang (2017)]
- Causality and inference
[Jacobs, Stein (2023); Jacobs, Zamani, Kissinger (2021)]

⋮

A CATEGORICAL STRUCTURE FOR PROBABILISTIC UPDATES

- Find the algebraic structure to express updates on probabilistic models.



PARTIAL MARKOV CATEGORIES

A partial Markov category is a copy-discard category where every morphism has a conditional.

→ copy-discard structure, $A \multimap$ and $A \rightarrow$, satisfying

$$\begin{array}{c} \text{---} \bullet \text{---} \\ \text{---} \bullet \end{array} = \text{---} \quad \begin{array}{c} \text{---} \bullet \\ \text{---} \bullet \end{array} = \begin{array}{c} \text{---} \bullet \\ \text{---} \bullet \end{array} \quad \begin{array}{c} \text{---} \bullet \\ \text{---} \bullet \end{array} = \begin{array}{c} \text{---} \bullet \end{array}$$

→ every morphism, $f: X \rightarrow A \otimes B$, has a conditional, $c: A \otimes X \rightarrow B$

$$X \text{---} \boxed{f} \text{---} \begin{array}{c} A \\ B \end{array} = X \text{---} \bullet \begin{array}{c} \boxed{f} \\ \bullet \end{array} \begin{array}{c} \text{---} A \\ \text{---} \bullet \end{array} \boxed{c} \text{---} B$$

EXAMPLES

- Stoch and Stoch_≤
- Set and Par (trivially)
- Rel
- BorelStoch_≤: measurable substochastic kernels on standard Borel spaces

$$f: (X, \sigma) \longrightarrow (A, \tau) \text{ is } f: \tau \times X \longrightarrow [0, 1]$$

$$f(S|x) = \text{"probability of event } S \text{ given input } x \text{"}$$

- Gauss and Gauss_≤: (partial) gaussian kernels
 $f: m \rightarrow m$ is $(M_g \in \mathbb{R}^{m \times m}, C_g \in \mathbb{R}^{m \times m}, \mu_g \in \mathbb{R}^m)$, C_g positive semidefinite
 $Y = M_g \cdot X + z \quad z \sim N(\mu_g, C_g)$

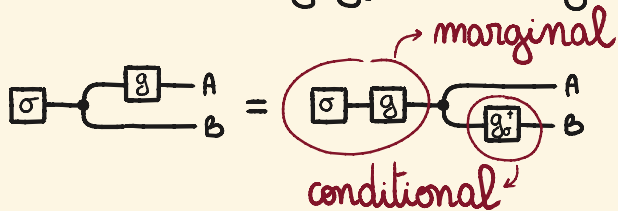
[Jutz (2020); EDL, Román, Sobociński, Széles (2025)]

BAYESIAN INVERSION

The bayesian inversion of a channel $g: B \rightarrow A$ with respect to a distribution $\sigma: I \rightarrow B$ is classically defined as

$$g_{\sigma}^{\dagger}(b|a) := \frac{g(a|b) \sigma(b)}{\sum_{b' \in B} g(a|b') \sigma(b')}$$

In a partial Markov category, it is a $g_{\sigma}^{\dagger}: A \rightarrow B$ such that



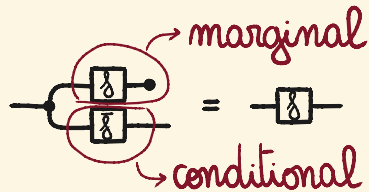
Bayes inversions are instances of conditionals.

NORMALISATION

The normalisation of a partial channel $f: X \rightarrow A$ is classically defined as

$$\bar{f}(a|x) := \frac{f(x|a)}{1 - f(\perp|a)}$$

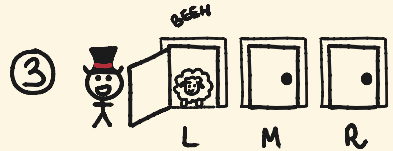
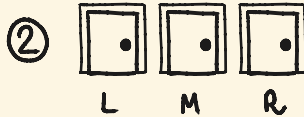
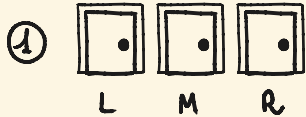
In a partial Markov category, it is a $\bar{f}: X \rightarrow A$ such that



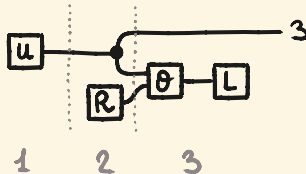
Normalisations are instances of conditionals.

[EDL, Román (2023); EDL, Román, Sobociński (2025)]

EXAMPLE : MONTY-HALL

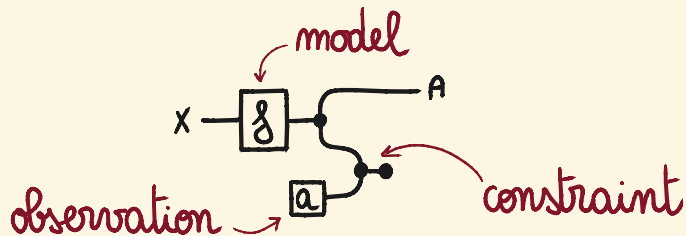


Should you switch?



PARTIALITY FOR OBSERVATIONS

Updating a model on an observation means restricting the model to scenarios that are compatible with this observation.

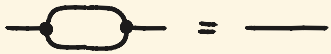


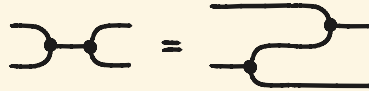
constraints $\curvearrowright$ cannot be total computations
because $\curvearrowright \neq \equiv$.

DISCRETE PARTIAL MARKOV CATEGORIES

A discrete partial Markov category is a copy-discard category with conditionals and compare morphisms.

$\rightarrow$ A satisfying the partial Frobenius axioms


$$\text{loop} = \text{line}$$

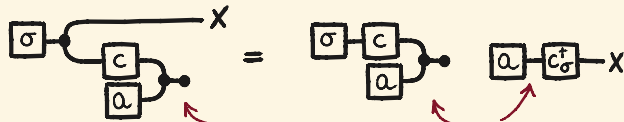

$$\text{cross} = \text{cross}$$

$\leadsto$ it behaves like an equality check

[EDL, Román (2023); EDL, Román, Sobociński (2025)]

SYNTHETIC BAYES THEOREM

A deterministic observation $a: I \rightarrow A$ from a prior $\sigma: I \rightarrow X$ through a channel $c: X \rightarrow A$ determines an update proportional to the Bayes inversion c_σ^+ evaluated on a .



$$P(X=x|A=a) = \frac{P(A=a|X=x) \cdot P(X=x)}{\sum_{y \in X} P(A=a|X=y) \cdot P(X=y)}$$

classical formula
for Bayes theorem

SYNTHETIC BAYES THEOREM

A deterministic observation $a: I \rightarrow A$ from a prior $\sigma: I \rightarrow X$ through a channel $c: X \rightarrow A$ determines an update proportional to the Bayes inversion c_σ^+ evaluated on a .

$$\sigma \rightarrow \begin{array}{c} c \\ a \end{array} \rightarrow X = \sigma \rightarrow c \rightarrow a \rightarrow X = a \rightarrow c_\sigma^+ \rightarrow X$$

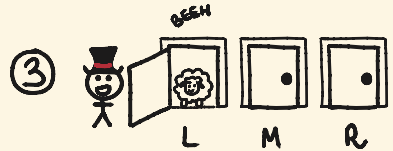
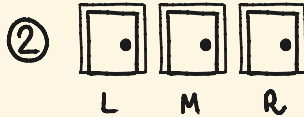
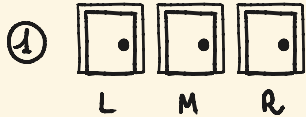
PROOF

$$\sigma \rightarrow \begin{array}{c} c \\ a \end{array} \rightarrow X = \sigma \rightarrow c \rightarrow a \rightarrow X = \sigma \rightarrow c \rightarrow a \rightarrow X = \sigma \rightarrow c \rightarrow a \rightarrow X$$

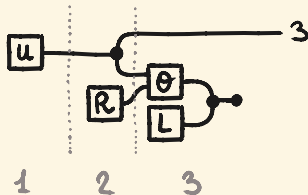
conditionals Frobenius determinism

□

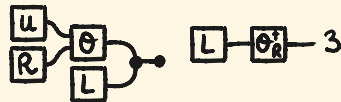
EXAMPLE: MONTY-HALL, CONTINUED



Should you switch? YES



(BAYES)
=

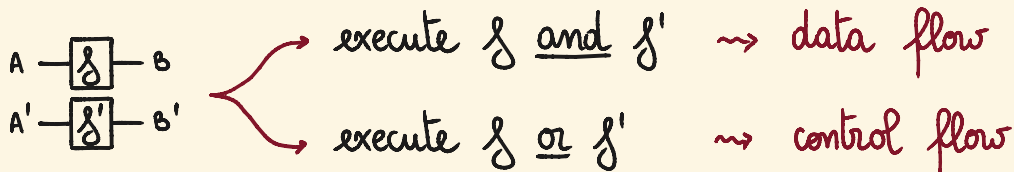


SUMMARY SO FAR

- Monoidal categories as algebras of processes/programs
- copy-discard, Markov and partial Markov categories as algebras of probabilistic processes/programs

→ there is much more: tailor your categorical structure depending on which phenomena the categorical model should capture

MORE MONOIDAL STRUCTURES: DATA VS CONTROL FLOW



ex $(\text{Stoch}_{\leq}, \otimes, \{*\})$ vs $(\text{Stoch}_{\leq}, +, \emptyset)$

We need both kinds of parallel compositions
 $\Rightarrow$ distributive monoidal categories $(\mathcal{C}, \otimes, 1, +, 0)$

BACK TO PROGRAM SEMANTICS

Von Neumann's trick

unbiased (p)

$x := 0$

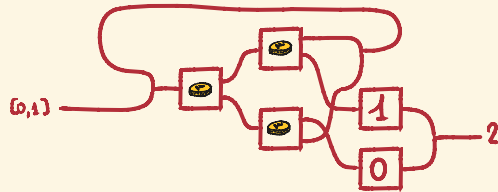
$y := 0$

while $x = y$

$x \leftarrow \text{coin}(p)$

$y \leftarrow \text{coin}(p)$

return x



We need traced distributive copy-discard categories.

[cf. Bonchi, EN, Román, Staton (2026)]

PREMONOIDAL CATEGORIES FOR NONCOMMUTATIVE EFFECTS

Effectful computations may have global effects.



→ premonoidal categories are monoidal categories that lack the interchange axiom

ex Kleisli categories of state monads and writer monads

[Power and Robinson (1997)]

SUMMARY AND CONCLUSIONS

- Categorical structures are good modelling languages
 - ↳ abstraction and compositionality for conceptual understanding
 - precise tailoring of the axioms to specific phenomena
- Monoidal and copy-discard categories as algebras of processes/programs
- Partial Markov categories support updating on observations.

The diagram illustrates the Synthetic Bayes theorem. On the left, a square node σ has an output wire that splits: one branch goes to a square node c , and the other goes to a square node a . The output of c is a wire labeled x . The output of a is a dot. On the right, the expression is equal to a square node σ followed by a square node c . The output of c is a dot. This dot is connected to a square node a , which is then connected to a square node c^+ . The output of c^+ is a wire labeled x .

Synthetic Bayes theorem

DISTRIBUTIVE COPY-DISCARD CATEGORIES

A distributive copy-discard category is a copy-discard category with (finite) coproducts such that the canonical morphisms

$$(A \otimes B_1) + (A \otimes B_2) \longrightarrow A \otimes (B_1 + B_2)$$

$$0 \longrightarrow A \otimes 0$$

$$(A_1 \otimes B) + (A_2 \otimes B) \longrightarrow (A_1 + A_2) \otimes B$$

$$0 \longrightarrow 0 \otimes A$$

are isomorphisms.