SYCO 8 - Tallinn

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MONOIDAL WIDTH

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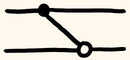
MOTIVATION

- measure of complexity for morphisms in a monoidal category
- find a common general framework for different measures of complexity for graphs

MAIN IDEA

- bring the ideas from graph complexity to monoidal categories
- measure the cost of decomposing a morphism into basic building blocks using some chosen operations

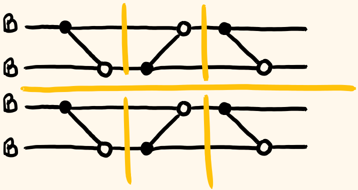
MAIN IDEA : EXAMPLE

Generators = {  }

CNOT gate

Operations = { composition, monoidal product }

A monoidal decomposition of exchanging the values of two pairs of variables.



MAIN RESULTS

- definition of monoidal width for measuring complexity of morphisms
- recover some known notions of complexity for graphs: path width, tree width, branch width, rank width

OUTLINE

- monoidal decompositions
- monoidal width for matrices
- monoidal width for graphs

DECOMPOSITION SYSTEMS

$(\mathcal{C}, \mathcal{G}, \mathcal{O}, w)$ decomposition system if

- $\mathcal{C}$ monoidal category
- $\mathcal{G}$ set of generators of $\mathcal{C}$
- $\mathcal{O} = \{\otimes, ;_X$ for any object $X\}$ set of operations
- $w : \mathcal{G} \cup \mathcal{O} \longrightarrow \mathbb{N}$ weight function such that

$$\begin{cases} w(\otimes) = 0 \\ w(;_{X\otimes Y}) = w(;_X) + w(;_Y) \end{cases}$$

DECOMPOSITION SYSTEMS - EXAMPLE

$(\mathcal{C}, \mathcal{G}, \mathcal{O}, w)$

- $\mathcal{C}$ monoidal category $\rightsquigarrow$ $\mathcal{C} := \mathcal{F}in\mathcal{S}et$
- $\mathcal{G}$ set of generators of $\mathcal{C}$ $\rightsquigarrow$ $\mathcal{G} := \{ -, \supset\!-, \bullet\!-, \times \}$
- $\mathcal{O} = \{ \otimes, ;_X$ for any object $X \}$ set of operations
- $w : \mathcal{G} \cup \mathcal{O} \longrightarrow \mathbb{N}$ weight such that

$$\begin{cases} w(\otimes) = 0 \\ w(;_{X \otimes Y}) = w(;_X) + w(;_Y) \end{cases}$$

$\rightsquigarrow$

$w(-) := 1$
$w(\supset\!-) := 2$
$w(\bullet\!-) := 1$
$w(\times) := 2$
$w(;_n) := n$

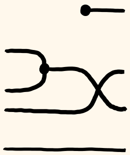
MONOIDAL DECOMPOSITIONS

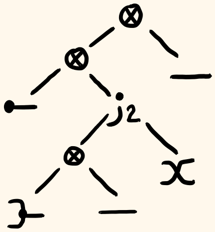
$f : X \longrightarrow Y$ morphism in $\mathcal{C}$

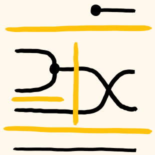
(S, μ) monoidal decomposition of f if

- S is a binary tree
- $\mu : \begin{cases} \text{internal nodes}(S) \longrightarrow \mathcal{O} \\ \text{leaves}(S) \longrightarrow \mathcal{G} \end{cases}$ labelling of S
- f is obtained by assembling the generators on the leaves of S according to the operations on the internal nodes of S

MONOIDAL DECOMPOSITIONS – EXAMPLE

 $: 4 \longrightarrow 4$ morphism in FinSet

$(S, \mu) =$ 

$\rightsquigarrow$ 

MONOIDAL WIDTH

(S, μ) monoidal decomposition of $f : X \to Y$

WIDTH OF A MONOIDAL DECOMPOSITION

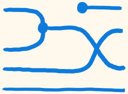
$$wd(S,\mu) := \max_{n \in nodes(S)} w(\mu(n))$$

MONOIDAL WIDTH

$$mwd(f) := \min_{(S,\mu)} wd(S,\mu)$$

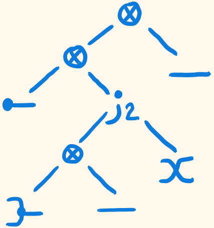
MONOIDAL WIDTH - EXAMPLE

(S,μ) monoidal decomposition of $f : X \to Y$

$f =$ 

WIDTH OF A MONOIDAL DECOMPOSITION

$$wd(S,\mu) := \max_{n \in nodes(S)} w(\mu(n))$$

$wd\Big($ 

$\Big) = 2$

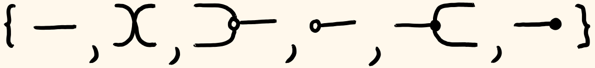
MONOIDAL WIDTH

$$mwd(f) := \min_{(S,\mu)} wd(S,\mu)$$

OUTLINE

- monoidal decompositions
- [monoidal width for matrices]
- monoidal width for graphs

MONOIDAL WIDTH OF MATRICES

$\mathcal{G} =$ 

$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & A_k \end{pmatrix} = A_1 \oplus A_2 \oplus \cdots \oplus A_k$$

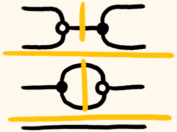
THEOREM

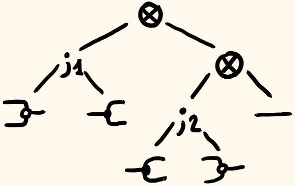
$$\max_i \operatorname{rank} A_i \leq \operatorname{mwd} A \leq \max_i \operatorname{rank} A_i + 1$$

MONOIDAL WIDTH OF MATRICES - EXAMPLE

$\mathcal{G} =$ 

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$





THEOREM

$$\max_i \operatorname{rank} A_i \leq \operatorname{mwd} A \leq \max_i \operatorname{rank} A_i + 1$$

OUTLINE

- monoidal decompositions
- monoidal width for matrices
- monoidal width for graphs

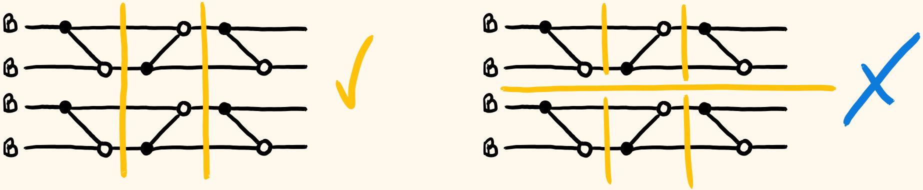
VARIATIONS OVER MONOIDAL WIDTH (1)

$(\mathcal{C}, \mathcal{G}, \mathcal{O}, w)$ decomposition system

generators of $\mathcal{C}$ allowed operations weight function

CATEGORICAL PATH WIDTH

$\mathcal{O} = \{ ;_X$ for any object $X \}$



VARIATIONS OVER MONOIDAL WIDTH (2)

$(\mathcal{C}, \mathcal{G}, \mathcal{O}, w)$ decomposition system

generators of $\mathcal{C}$

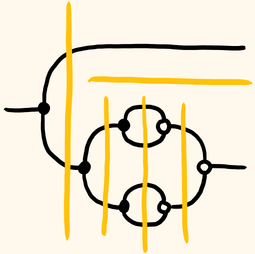
allowed operations

weight function

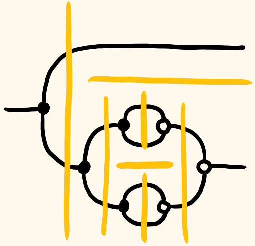
CATEGORICAL TREE WIDTH

$\mathcal{O} = \{\otimes, ;_X$ for any object $X\}$

composition restricted to generators on the left



✓



✗

TREE WIDTH [Robertson & Seymour, 1986]

$G = (V, E)$ undirected graph

TREE DECOMPOSITION

(T, λ) where

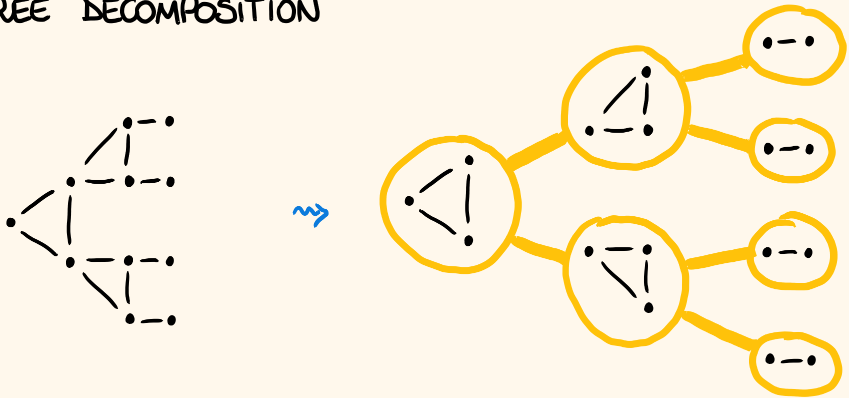
- T binary tree
- $\lambda : \text{nodes}(T) \longrightarrow \text{subgraphs}(G)$ such that

$$\begin{cases} \bigcup_{n \in \text{nodes}(T)} \lambda(n) = G \\ \forall\, n \in \text{nodes}(T) \quad \lambda(n) \supseteq \lambda(n.\text{left}) \cap \lambda(n.\text{right}) \end{cases}$$

TREE WIDTH - EXAMPLE

$G = (V, E)$ undirected graph

TREE DECOMPOSITION

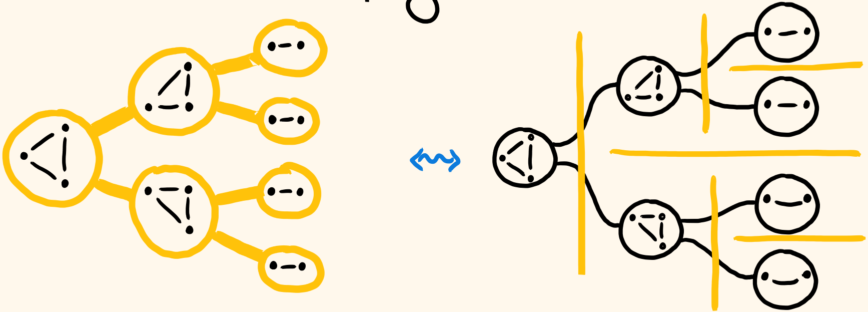


TREE WIDTH & MONOIDAL WIDTH

$G = (V, E)$ undirected graph
$g = \emptyset \to G \leftarrow \emptyset$ cospan corresponding to G

THEOREM

$$\mathrm{twd}\, G \leq \mathrm{mwd}_T\, g \leq 2 \cdot \mathrm{twd}\, G$$



SUMMARY OF RESULTS

• matrices	mwd	$\leftrightsquigarrow$	rank
• cospans of graphs	mwd_T	$\leftrightsquigarrow$	tree width
	mwd_P	$\leftrightsquigarrow$	path width
	mwd	$\leftrightsquigarrow$	branch width
• prop of graphs	mwd	$\leftrightsquigarrow$	rank width

FUTURE WORK

- monoidal width in other categories
- directed tree width, DAG width, Kelly width, cut width, clique width, ...
- monoidal width 'functorially' (describe the weight function as a functor)

THANKS ! 

SOME REFERENCES

GRAPH WIDTHS

- Robertson & Seymour, Graph minors I-X, 1983-1991
- Oum & Seymour, Approximating clique width and branch width, 2006
- Johnson, Robertson, Seymour & Thomas, Directed tree width, 2001
- Courcelle & Olariu, Upper bounds to the clique width of graphs, 2000
- Hunter & Kreutzer, Kelly decompositions, games and orderings, 2008
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  (Di Lavore, Hedges, Sobociński, Compositional modelling of network games, 2021)