# MONOIDAL WIDTH

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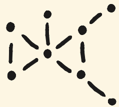
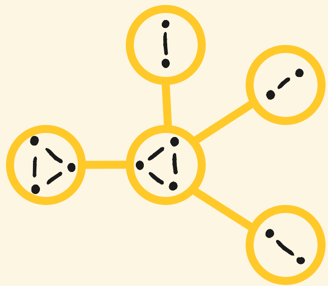
# FIXED-PARAMETER TRACTABILITY

Some problems might be easier to solve on structurally "simple" inputs.

**THEOREM (Courcelle 1990)**

Every property expressible in the monadic second order logic of graphs can be verified in linear time on graphs of bounded tree width.

$G =$ 

$(Y, t) =$ 

# OVERVIEW

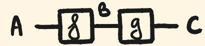
- Monoidal categories give process theories.
- Study fixed-parameter tractability of problems on morphisms in monoidal categories.
- Introduce monoidal width to measure structural complexity in monoidal categories.
- Capture tree width and rank width.

# STRING DIAGRAMS

$\mathcal{C}$ symmetric monoidal category

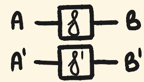
$f : A \to B$ , $g : B \to C$ in $\mathcal{C}$

- composition $f ; g : A \to C$

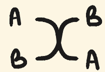


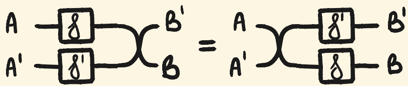

$f : A \to B$ , $f' : A' \to B'$ in $\mathcal{C}$

- monoidal product $f \otimes f' : A \otimes A' \to B \otimes B'$




- symmetry $\sigma_{A,B} : A \otimes B \to B \otimes A$




(naturality)

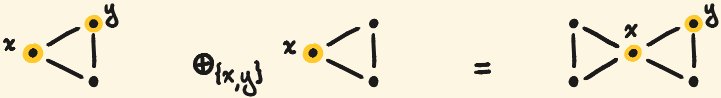
# OUTLINE

- Monoidal decompositions
- Matrices
- Rank width
- Branch width
- Fixed-parameter tractability

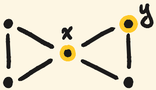
# TREE DECOMPOSITIONS

Operation $\oplus_X$ on graphs with sources:

$G \oplus_X H$ glues $G$ and $H$ along the sources in $X$.




A <u>tree decomposition</u> of $G$ is a term for $G$, where operations are gluing along sources $\oplus_X$, deletion of sources $\varepsilon_X$ and an edge generator $e_{xy}$.




$$= \varepsilon_{\{z\}}((e_{xy} \oplus_{\{x,y,z\}} e_{yz}) \oplus_{\{x,z\}} e_{xz})$$
$$\oplus_{\{x,y\}} \varepsilon_{\{y,z\}}((e_{xy} \oplus_{\{x,y,z\}} e_{yz}) \oplus_{\{x,z\}} e_{xz})$$

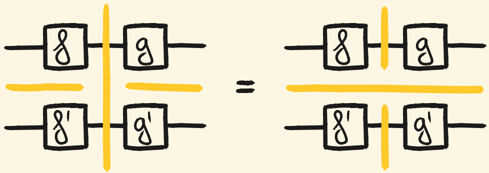
[Robertson & Seymour 1983, Courcelle 1990]

# DECOMPOSING MORPHISMS IN MONOIDAL CATEGORIES

There are two operations in monoidal categories:

- composition $;$ → resource sharing, synchronisation
  ⇒ COSTLY
- monoidal product $\otimes$ → processes side-by-side
  ⇒ CHEAP

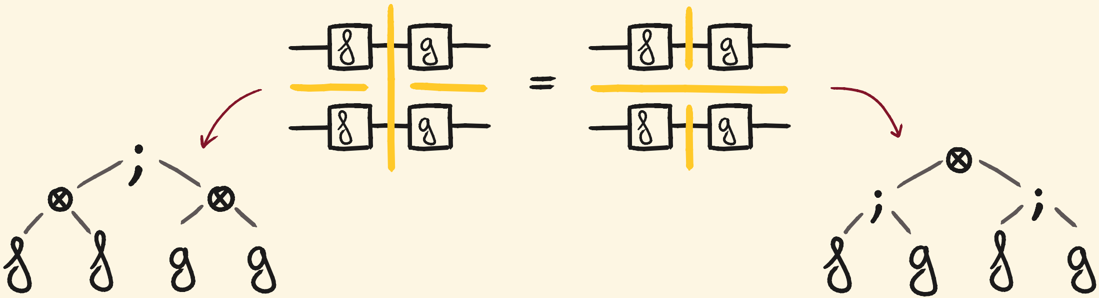
$$(f \otimes f') ; (g \otimes g') = (f ; g) \otimes (f' ; g')$$



# MONOIDAL DECOMPOSITIONS

A monoidal decomposition $d \in \mathcal{D}_f$ of $f : X \to Y$ is

$$
\begin{aligned}
d \; ::= \; & (f) \\
\mid \; & d_1 \mathbin{;_c} d_2 \quad \text{if } f = f_1 \mathbin{;_c} f_2,\; d_1 \in \mathcal{D}_{f_1},\; d_2 \in \mathcal{D}_{f_2} \\
\mid \; & d_1 \otimes d_2 \quad \text{if } f = f_1 \otimes f_2,\; d_1 \in \mathcal{D}_{f_1},\; d_2 \in \mathcal{D}_{f_2}
\end{aligned}
$$



# MONOIDAL WIDTH

## WEIGHT FUNCTION

$w : \mathrm{morph}\,\mathcal{C} \to \mathbb{N}$ such that

- $w(f ;_Y g) + w(\mathbb{1}_Y) \geq w(f) + w(g)$
- $w(f \otimes g) = w(f) + w(g)$

## WIDTH OF A DECOMPOSITION

⇝ cost of the most expensive operation

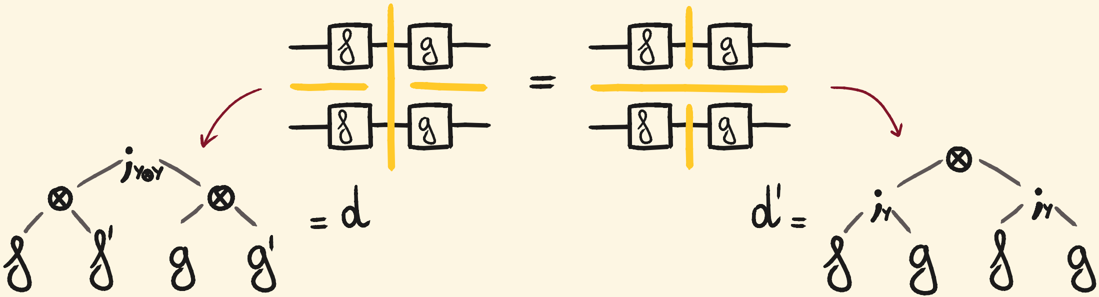
$$
\begin{aligned}
wd(d) := \; & w(f) && d = (f) \\
\mid \; & \max\{wd(d_1), w(\mathbb{1}_Y), wd(d_2)\} && d = d_1 \overset{;_Y}{\wedge} d_2 \\
\mid \; & \max\{wd(d_1), wd(d_2)\} && d = d_1 \overset{\otimes}{\wedge} d_2
\end{aligned}
$$

## MONOIDAL WIDTH

$$mwd(f) := \min_{d \in \mathcal{D}_f} wd(d)$$

cost of a cheapest decomposition

# MONOIDAL WIDTH INCENTIVISES PARALLELISM

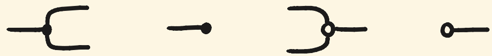



$$\mathrm{wd}(d) = \max\{w(f), w(g), 2 \cdot w(\mathbb{1}_Y)\} \geq \max\{w(f), w(g), w(\mathbb{1}_Y)\} = \mathrm{wd}(d')$$

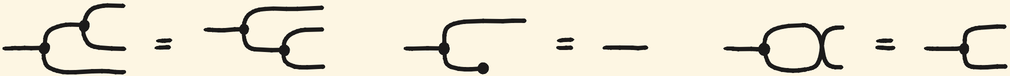
# OUTLINE

- Monoidal decompositions
- [ Matrices ]
- Rank width
- Branch width
- Fixed-parameter tractability

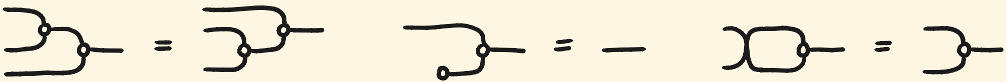
# BIALGEBRA : THE PROP OF $\mathbb{N}$ - MATRICES



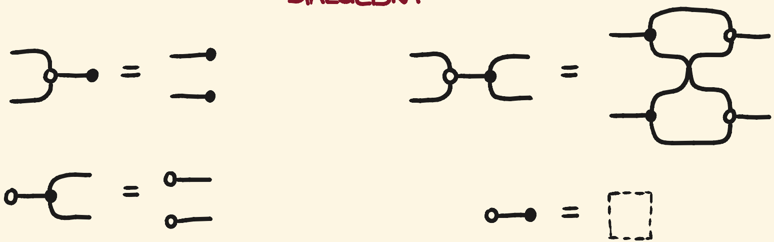
## COCOMMUTATIVE COMONOID



## COMMUTATIVE MONOID



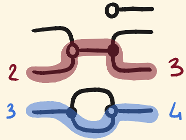
## BIALGEBRA



[Zanasi 2015]

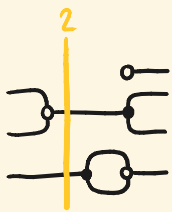
# PROP OF MATRICES - EXAMPLE

$$A = \begin{array}{c|c} & \begin{array}{ccc} & 2 & 3 \end{array} \\ \begin{array}{c} \\ \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \end{array} =$$



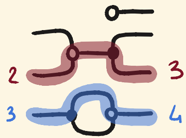

FACT : the minimal vertical cut in a matrix is its rank : $\min\{k \in \mathbb{N} \mid A = B ;_k C\} = \operatorname{rank} A$

$\operatorname{rank} A = 2 \rightsquigarrow$



# PROP OF MATRICES – EXAMPLE

$$A = \begin{array}{rc} & \begin{array}{ccc} \phantom{0} & \downarrow^{2} & \downarrow^{3} \end{array} \\ \begin{array}{r} \\ \\ 3 \rightarrow \\ 4 \rightarrow \end{array} & \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \end{array} =$$

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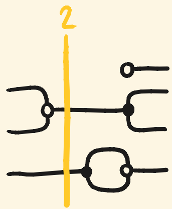
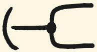
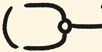


# THE WIDTH OF NATURAL NUMBERS

$$\begin{aligned} w : \mathrm{morph}(\mathrm{Bialg}) &\longrightarrow \mathbb{N} \\ f : n \to m &\longmapsto \max\{m, n\} \end{aligned}$$

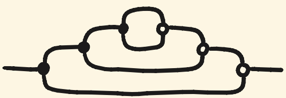
$\rightsquigarrow$ $w($  $) = 2$

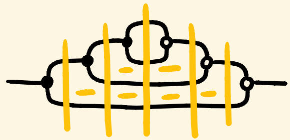
$w($  $) = 2$

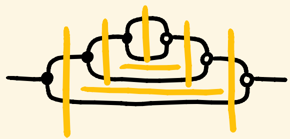
LEMMA

$$\mathrm{mwd}((n)) \leq 2$$

<u>ex</u>

$4 =$ 

$\mathrm{wd}($  $) = 4$

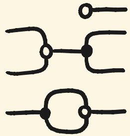
$\mathrm{wd}($  $) = 2$

# THE WIDTH OF MATRICES

Matrices can be written in blocks:

$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & A_b \end{pmatrix} = A_1 \oplus A_2 \oplus \cdots \oplus A_b$$

<u>ex</u> $A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} =$

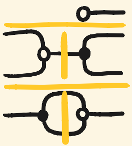


$= \mathrm{i} \oplus \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \oplus (2)$

THEOREM

$$\max_i \operatorname{rank} A_i \;\le\; \operatorname{mwd} A \;\le\; \max_i \operatorname{rank} A_i + 1$$

<u>ex</u> $\operatorname{wd}\Bigg($



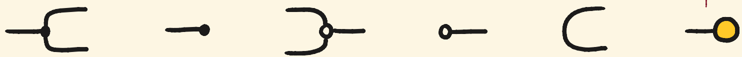
$\Bigg) = \max\left\{ \underset{\overset{\shortparallel}{0}}{r(\mathrm{i})},\ \underset{\overset{\shortparallel}{1}}{r\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}},\ \underset{\overset{\shortparallel}{1}}{r(2)} \right\} + 1 = 2$

# OUTLINE

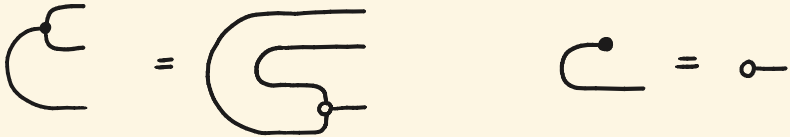
- Monoidal decompositions
- Matrices
- Rank width
- Branch width
- Fixed-parameter tractability

# A PROP OF GRAPHS

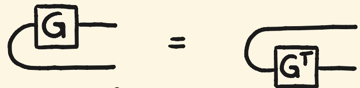
vertex generator



bialgebra equations +



⇝ the cup transposes

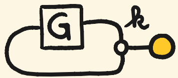



and captures equivalence of adjacency matrices

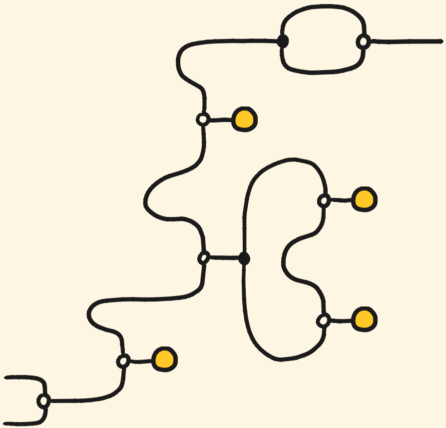
$[G] = [H] \iff$


[Di Lavore, Hedges & Sobociński 2021]

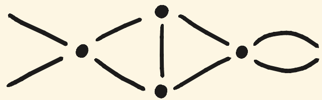
# GRAPHS AS MORPHISMS - EXAMPLE




$\rightsquigarrow$ graph on $k$ vertices given by the adjacency matrix $[G]$



$\rightsquigarrow$



# RANK WIDTH [Oum & Seymour, 2006]

G undirected graph

## RANK DECOMPOSITION

$(Y, \pi)$ where

- $Y$ is a subcubic tree (= any node has at most 3 neighbours)
- $\pi$ : leaves$(Y) \xrightarrow{\simeq}$ vertices$(G)$ labelling bijection

## WIDTH OF $(Y, \pi)$

$$\mathrm{wd}(Y, \pi) := \max_{e \in \mathrm{edges}\,Y} \mathrm{rank}(X_e)$$

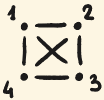
$X_e$ adjacency matrix of the cut given by e through $\pi$

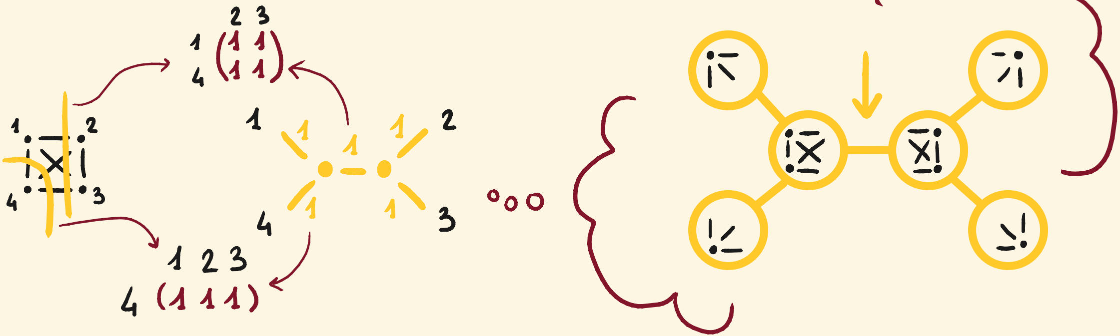
## RANK WIDTH

$$\mathrm{rwd}(G) := \min_{(Y, \pi)} \mathrm{wd}(Y, \pi)$$

⇝ cost of a cheapest decomposition

# RANK WIDTH - EXAMPLE

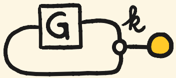
$G =$ 


$rwd(G) = 1$

# RANK WIDTH & MONOIDAL WIDTH

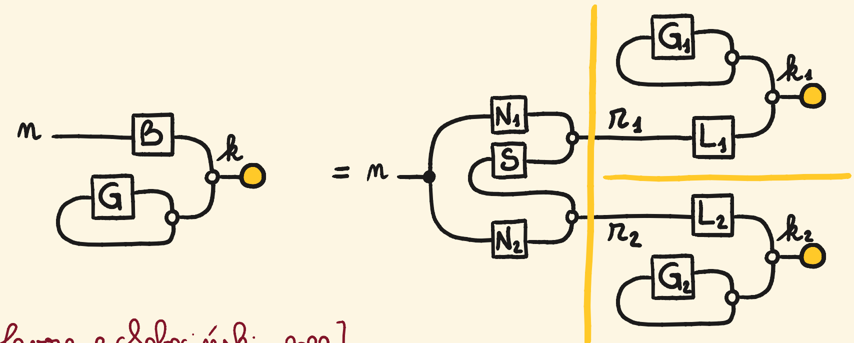
[G] undirected graph

$g =$ 

$: 0 \to 0$ in Grph

THEOREM

$$\frac{1}{2}\,\mathrm{rwd}(G) \leq \mathrm{mwd}(g) \leq 2\,\mathrm{rwd}(G)$$




[Di Lavore & Sobociński 2022]

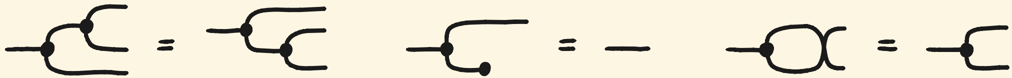
# OUTLINE

- Monoidal decompositions
- Matrices
- Rank width
- [ Branch width ]
- Fixed-parameter tractability

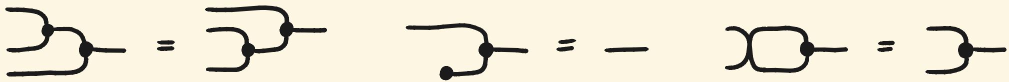
# FROBENIUS : A PROP OF τ-STRUCTURES

for $R \in \tau$ of arity $n$

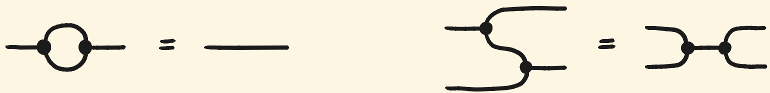
## COCOMMUTATIVE COMONOID



## COMMUTATIVE MONOID



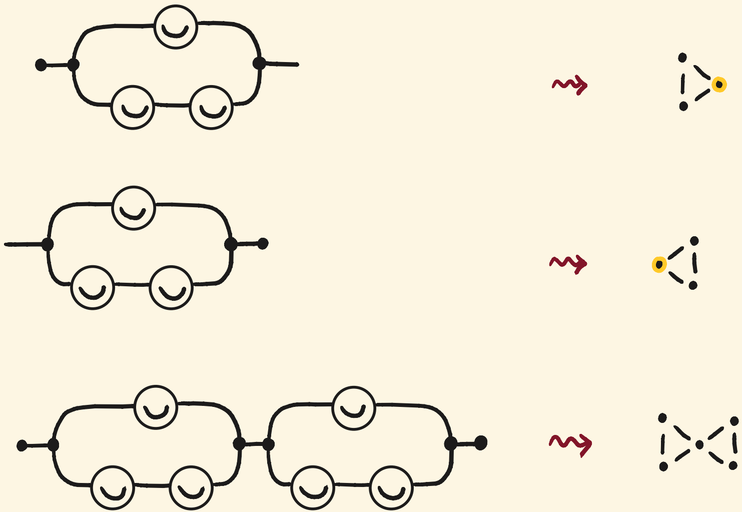
## FROBENIUS

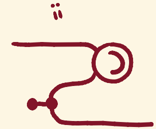


[Bonchi, Seeber & Sobociński 2018]

# GRAPHS AS MORPHISMS – EXAMPLE

graphs with sources are $\tau$-structures with $\tau = \{ \quad \}$





# BRANCH WIDTH [Robertson & Seymour, 1991]

$G$ undirected graph

## BRANCH DECOMPOSITION

$(Y, b)$ where

- $Y$ is a subcubic tree ( = any node has at most 3 neighbours)
- $b : \mathrm{leaves}(Y) \xrightarrow{\simeq} \mathrm{edges}(G)$ labelling bijection

## WIDTH OF $(Y, b)$

$$\mathrm{wd}(Y,b) := \max_{e \in \mathrm{edges}\,Y} |\mathrm{ends}\,A_e \cap \mathrm{ends}\,B_e|$$

$\{A_e, B_e\}$ partition of $E$ given by $e$ through $b$

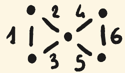
## BRANCH WIDTH

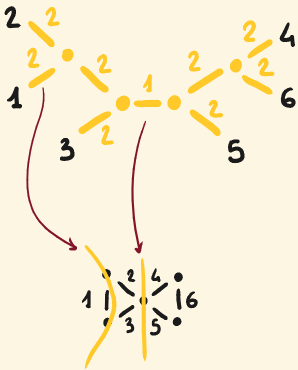
$$\mathrm{bwd}(G) := \min_{(Y,b)} \mathrm{wd}(Y,b)$$

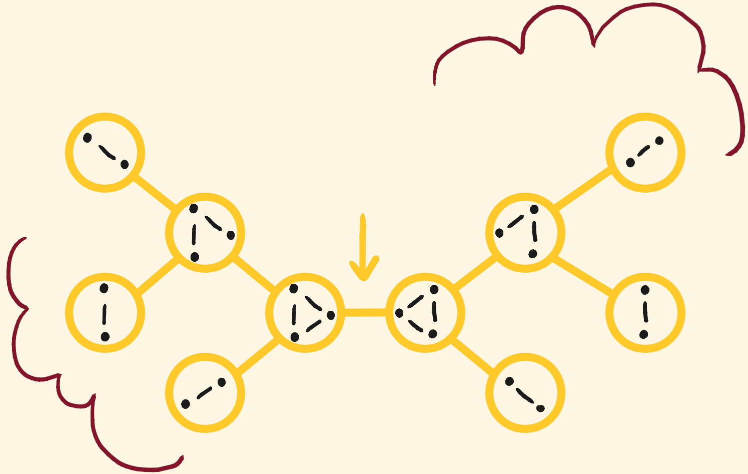
$\rightsquigarrow$ cost of a cheapest decomposition

# BRANCH WIDTH – EXAMPLE

G =



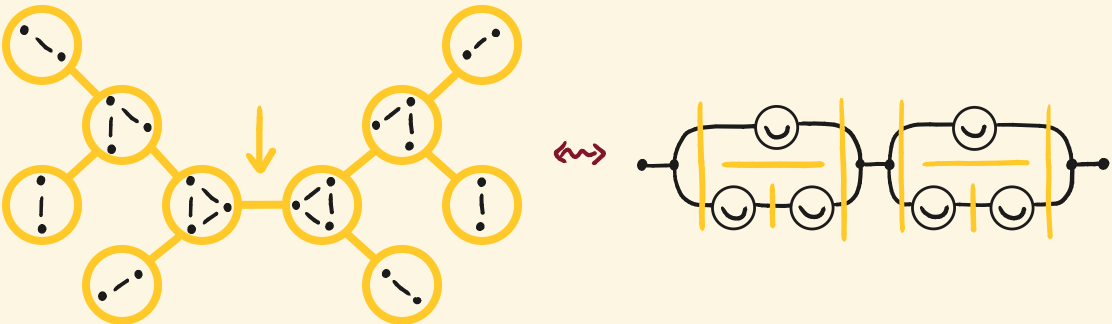



# BRANCH WIDTH & MONOIDAL WIDTH

$G = (V, E)$ undirected graph
$g = \varnothing \xrightarrow{i} G \xleftarrow{i} \varnothing : \varnothing \to \varnothing$ in $\mathrm{cospan}(\mathrm{UGraph})_{\varnothing}$

THEOREM

$$\frac{1}{2}\,\mathrm{bwd}(G) \leq \mathrm{mwd}(g) \leq \mathrm{bwd}(G) + 1$$



[Di Lavore & Sobociński 2022]

# OUTLINE

- Monoidal decompositions
- Matrices
- Rank width
- Branch width
- [ Fixed-parameter tractability ]

# COMPOSITIONAL ALGORITHMS

$\mathcal{C}, \mathcal{D}$ monoidal categories

$P : \mathcal{C} \to \mathcal{D}$ monoidal functor

space of solutions

$w : \mathrm{morph}\, \mathcal{C} \to \mathbb{N}$ weight function

cf. MSOL-smooth operations and MSOL-inductive classes of $\tau$-structures

cf. number of vertices

a <u>compositional algorithm</u> for $P$ w.r.t. $w$ computes

1. $P(f)$ in time $\mathcal{O}(c(w(f)) \cdot w(f))$
2. $P(f) ;_Y P(g)$ in $\mathcal{D}$ in time $\mathcal{O}(c(w(\mathbb{1}_Y)) \cdot (w(f) + w(g)))$
3. $P(f) \otimes P(f')$ in $\mathcal{D}$ in time $\mathcal{O}(c(1) \cdot (w(f) + w(f')))$

for some function $c : \mathbb{N} \to \mathbb{N}$.

usually more than exponential

[cf. Courcelle & Makowsky 2002]

# FEFERMAN-VAUGHT THEOREM

THEOREM

For $\tau$-structures $A, B, A', B'$ and a set $X$ of sources,
if $A \equiv_{MSO(\tau)} A'$ and $B \equiv_{MSO(\tau)} B'$, then $A \oplus_X B \equiv_{MSO(\tau)} A' \oplus_X B'$.
Computing $A \oplus_X B \models \varphi$ given $Th_{MSO_q(\tau)}(A)$ and $Th_{MSO_q(\tau)}(B)$ does not depend on $A \oplus_X B$.

COROLLARY

There is a monoidal functor

$$\begin{aligned} Th : Struct(\tau) &\longrightarrow Struct(\tau)/_{\equiv_{MSO(\tau)}} \\ A &\longmapsto Th_{MSO_q(\tau)}(A) \end{aligned}$$

that can be computed with a compositional algorithm.

[Feferman & Vaught 1959, Courcelle & Makowsky 2002]

# MONOIDAL FIXED-PARAMETER TRACTABILITY

THEOREM

Computing a functorial problem $P : \mathcal{C} \to \mathcal{D}$ with a compositional algorithm w.r.t. $w : \text{morph}\,\mathcal{C} \to \mathbb{N}$ is fixed-parameter tractable with parameter monoidal width.

Explicitly, if $\text{mwd}(g) \leq k$, computing $P(g)$ takes $\mathcal{O}(c(k) \cdot w(g))$, for some $c : \mathbb{N} \to \mathbb{N}$.

COROLLARY

Checking an MSO formula $\varphi \in MSO_q$ on $\tau$-structures is fixed-parameter tractable with parameter tree width.

# SUMMARY & FUTURE DIRECTIONS

- Monoidal width measures structural complexity of morphisms in monoidal categories.
- Monoidal width captures rank width and tree width.
- We would like to find other examples of fixed-parameter tractability more in the spirit of morphisms as processes.