Coresources workshop

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MONOIDAL WIDTH

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European Union
European Regional Development Fund

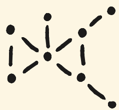
Investing in your future

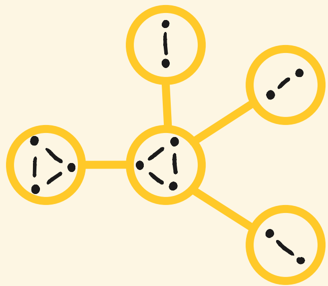
FIXED-PARAMETER TRACTABILITY

Some problems might be easier to solve on structurally "simple" inputs.

THEOREM (Courcelle 1990)

Every property expressible in the monadic second order logic of graphs can be verified in linear time on graphs of bounded tree width.

$G =$ 

$(Y, t) =$ 

OVERVIEW

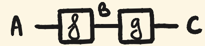
- Monoidal categories give process theories.
- Study fixed-parameter tractability of problems on morphisms in monoidal categories.
- Introduce monoidal width to measure structural complexity in monoidal categories.
- Capture tree width and rank width.

STRING DIAGRAMS

$\mathcal{C}$ symmetric monoidal category

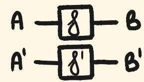
$f : A \to B$, $g : B \to C$ in $\mathcal{C}$

- composition $f ; g : A \to C$

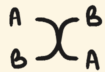


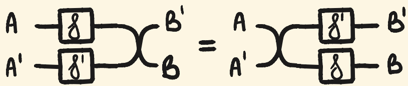
$f : A \to B$, $f' : A' \to B'$ in $\mathcal{C}$

- monoidal product $f \otimes f' : A \otimes A' \to B \otimes B'$



- symmetry $\sigma_{A,B} : A \otimes B \to B \otimes A$





(naturality)

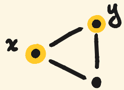
OUTLINE

- Monoidal decompositions
- Matrices
- Rank width
- Branch width
- Fixed-parameter tractability

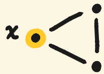
TREE DECOMPOSITIONS

Operation $\oplus_X$ on graphs with sources:

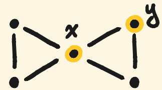
$G \oplus_X H$ glues G and H along the sources in X.



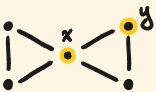
$\oplus_{\{x,y\}}$



$=$



A <u>tree decomposition</u> of G is a term for G, where operations are gluing along sources $\oplus_X$, deletion of sources ε_X and an edge generator e_{xy}.



$$= \varepsilon_{\{z\}}((e_{xy} \oplus_{\{x,y,z\}} e_{yz}) \oplus_{\{x,z\}} e_{xz})$$
$$\oplus_{\{x,y\}} \varepsilon_{\{y,z\}}((e_{xy} \oplus_{\{x,y,z\}} e_{yz}) \oplus_{\{x,z\}} e_{xz})$$

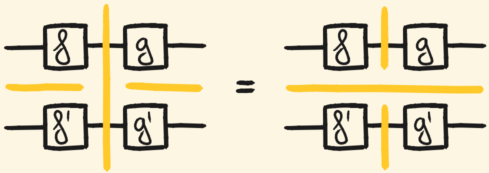
[Robertson & Seymour 1983, Courcelle 1990]

DECOMPOSING MORPHISMS IN MONOIDAL CATEGORIES

There are two operations in monoidal categories:

- composition $;$ ⇝ resource sharing, synchronisation
  ⇒ COSTLY
- monoidal product $\otimes$ ⇝ processes side-by-side
  ⇒ CHEAP

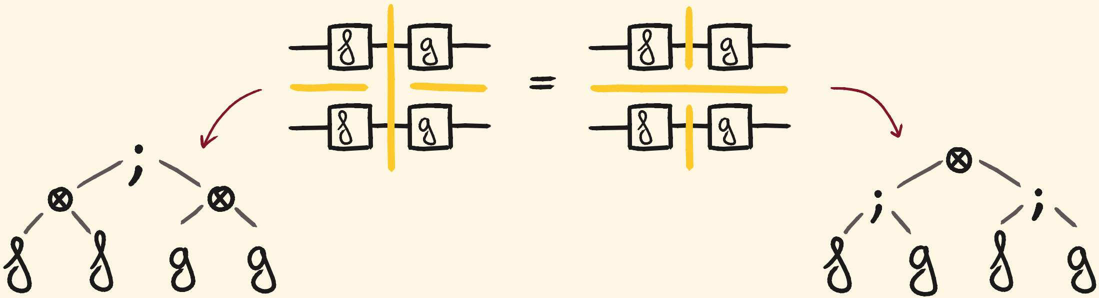
$$(f \otimes f') ; (g \otimes g') = (f ; g) \otimes (f' ; g')$$



MONOIDAL DECOMPOSITIONS

A monoidal decomposition $d \in \mathcal{D}_f$ of $f : X \to Y$ is

$$
\begin{aligned}
d \; ::= \; & (f) \\
\mid \; & d_1 \overset{;_c}{\frown} d_2 \quad \text{if } f = f_1 ;_c f_2 ,\ d_1 \in \mathcal{D}_{f_1} ,\ d_2 \in \mathcal{D}_{f_2} \\
\mid \; & d_1 \overset{\otimes}{\frown} d_2 \quad \text{if } f = f_1 \otimes f_2 ,\ d_1 \in \mathcal{D}_{f_1} ,\ d_2 \in \mathcal{D}_{f_2}
\end{aligned}
$$



MONOIDAL WIDTH

WEIGHT FUNCTION

$w : \mathrm{morph}\, \mathcal{C} \to \mathbb{N}$ such that

- $w(f ;_Y g) + w(\mathbb{1}_Y) \geq w(f) + w(g)$
- $w(f \otimes g) = w(f) + w(g)$

WIDTH OF A DECOMPOSITION

$\rightsquigarrow$ cost of the most expensive operation

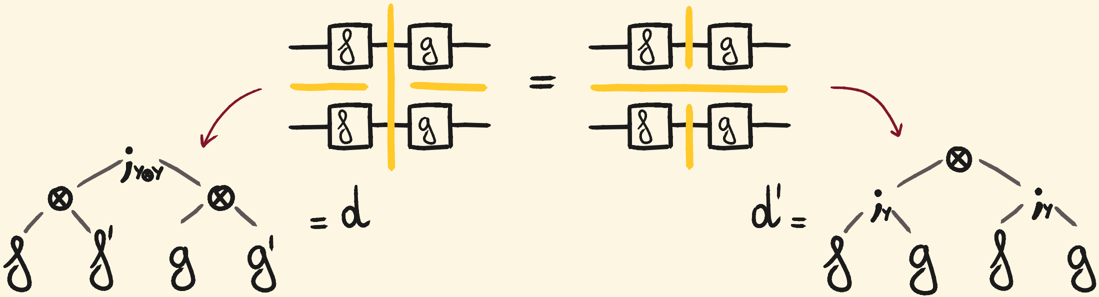
$$\begin{aligned} wd(d) := \; & w(f) & d = (f) \\ \mid \; & \max\{wd(d_1), w(\mathbb{1}_Y), wd(d_2)\} & d = d_1 \overset{;_Y}{\wedge} d_2 \\ \mid \; & \max\{wd(d_1), wd(d_2)\} & d = d_1 \overset{\otimes}{\wedge} d_2 \end{aligned}$$

MONOIDAL WIDTH

$$mwd(f) := \min_{d \in \mathcal{D}_f} wd(d)$$

cost of a cheapest decomposition

MONOIDAL WIDTH INCENTIVISES PARALLELISM

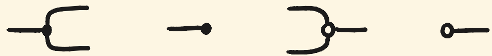


$$wd(d) = \max\{w(f), w(g), 2\cdot w(\mathbb{1}_Y)\} \geq \max\{w(f), w(g), w(\mathbb{1}_Y)\} = wd(d')$$

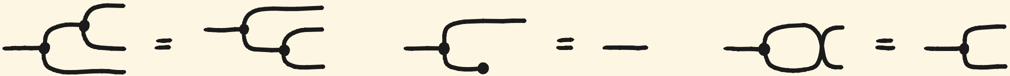
OUTLINE

- Monoidal decompositions
- [• Matrices]
- Rank width
- Branch width
- Fixed-parameter tractability

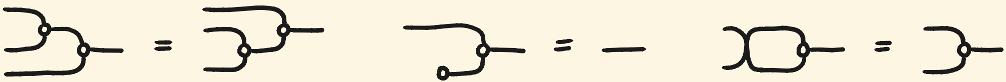
BIALGEBRA : THE PROP OF $\mathbb{N}$-MATRICES



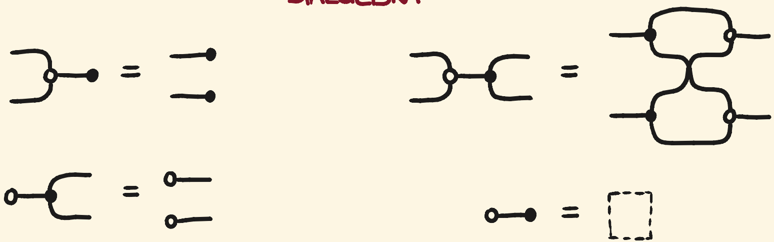
COCOMMUTATIVE COMONOID



COMMUTATIVE MONOID



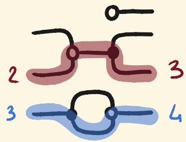
BIALGEBRA



[Zanasi 2015]

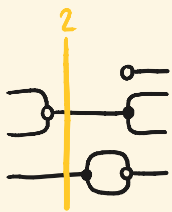
PROP OF MATRICES - EXAMPLE

$$A = \begin{array}{c c} & \begin{array}{ccc} & 2 & 3 \\ & \downarrow & \downarrow \end{array} \\ \begin{array}{c} \\ \\ 3 \rightarrow \\ 4 \rightarrow \end{array} & \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \end{array} =$$



FACT : the minimal vertical cut in a matrix is its rank : $\min\{k \in \mathbb{N} \mid A = B ;_k C\} = \operatorname{rank} A$

$\operatorname{rank} A = 2 \rightsquigarrow$



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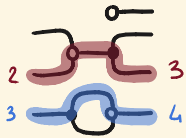

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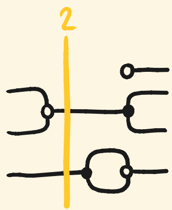
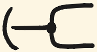
$\mathrm{rank}\, A = 2 \quad \rightsquigarrow$



THE WIDTH OF NATURAL NUMBERS

$$w : \mathrm{morph}(\mathrm{Bialg}) \longrightarrow \mathbb{N}$$
$$f : n \to m \longmapsto \max\{m, n\}$$

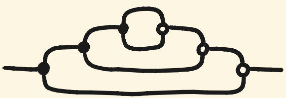
$\rightsquigarrow$ $w\Big($  $\Big) = 2$

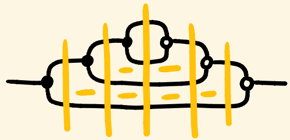
$w\Big($  $\Big) = 2$

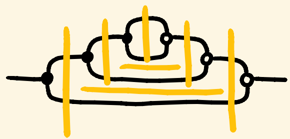
LEMMA

$$\mathrm{mwd}((n)) \leq 2$$

<u>ex</u>

$4 =$ 

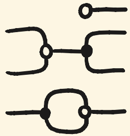
$\mathrm{wd}\Big($  $\Big) = 4$

$\mathrm{wd}\Big($  $\Big) = 2$

THE WIDTH OF MATRICES

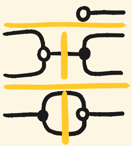
Matrices can be written in blocks :

$$A = \begin{pmatrix} A_1 & \mathbb{0} & \cdots & \mathbb{0} \\ \mathbb{0} & A_2 & & \vdots \\ \vdots & & \ddots & \\ \mathbb{0} & \cdots & & A_b \end{pmatrix} = A_1 \oplus A_2 \oplus \cdots \oplus A_b$$

ex $\quad A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} =$  $= i \oplus \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \oplus (2)$

THEOREM

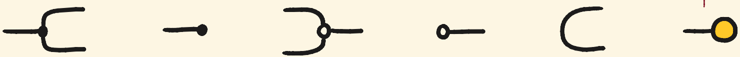
$$\max_i \operatorname{rank} A_i \leq \operatorname{mwd} A \leq \max_i \operatorname{rank} A_i + 1$$

ex $\quad \operatorname{wd}\Big($  $\Big) = \max\{\underset{0}{\underset{\parallel}{r(j)}}, \underset{1}{\underset{\parallel}{r\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}}, \underset{1}{\underset{\parallel}{r(2)}}\} + 1 = 2$

OUTLINE

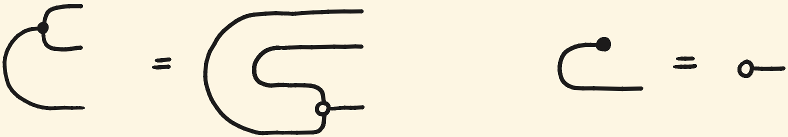
- Monoidal decompositions
- Matrices
- [Rank width]
- Branch width
- Fixed-parameter tractability

A PROP OF GRAPHS

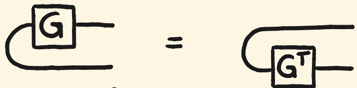


vertex generator

bialgebra equations +



⇝ the cup transposes

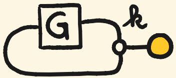


and captures equivalence of adjacency matrices

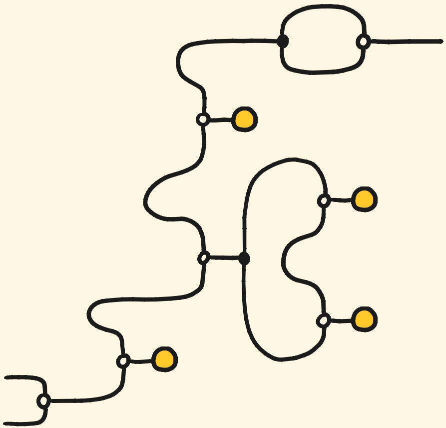
$[G] = [H] \iff$

[DL, Hedges & Sobociński 2021]

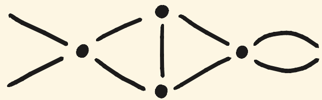
GRAPHS AS MORPHISMS – EXAMPLE



$\rightsquigarrow$ graph on k vertices given by the adjacency matrix $[G]$



$\rightsquigarrow$



RANK WIDTH [Oum & Seymour, 2006]

G undirected graph

RANK DECOMPOSITION

(Y, π) where

- Y is a subcubic tree (= any node has at most 3 neighbours)
- π : leaves$(Y) \xrightarrow{\simeq}$ vertices(G) labelling bijection

WIDTH OF (Y, π)

$$\mathrm{wd}(Y, \pi) := \max_{e \in \mathrm{edges}\, Y} \mathrm{rank}(X_e)$$

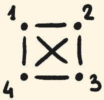
X_e adjacency matrix of the cut given by e through π

RANK WIDTH

$$\mathrm{rwd}(G) := \min_{(Y, \pi)} \mathrm{wd}(Y, \pi)$$

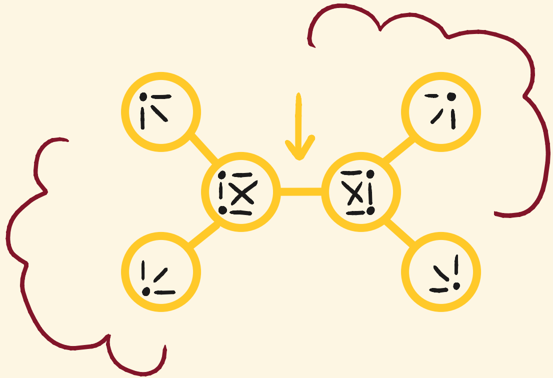
⇝ cost of a cheapest decomposition

RANK WIDTH - EXAMPLE

G = 

$$\begin{array}{c|cc} & 2 & 3 \\ \hline 1 & 1 & 1 \\ 4 & 1 & 1 \end{array}$$

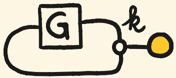
$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 4 & 1 & 1 & 1 \end{array}$$



rwd(G) = 1

RANK WIDTH & MONOIDAL WIDTH

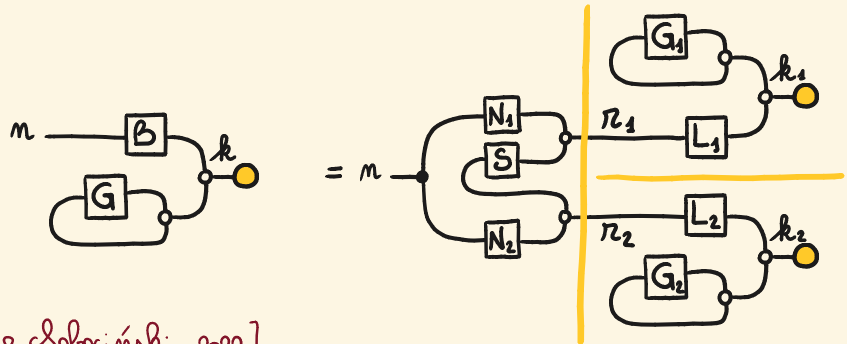
[G] undirected graph

$g =$ 

$: 0 \to 0$ in Grph

THEOREM

$$\frac{1}{2}\,\mathrm{rwd}(G) \leq \mathrm{mwd}(g) \leq 2\,\mathrm{rwd}(G)$$



[DL & Sobociński 2022]

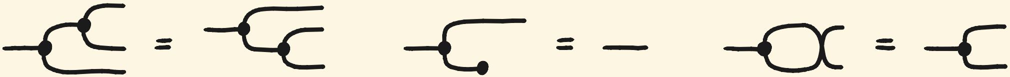
OUTLINE

- Monoidal decompositions
- Matrices
- Rank width
- [Branch width]
- Fixed-parameter tractability

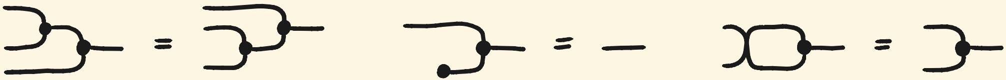
FROBENIUS : A PROP OF τ-STRUCTURES

for $R \in \tau$ of arity n

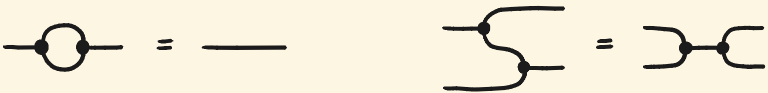
COCOMMUTATIVE COMONOID



COMMUTATIVE MONOID

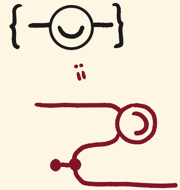


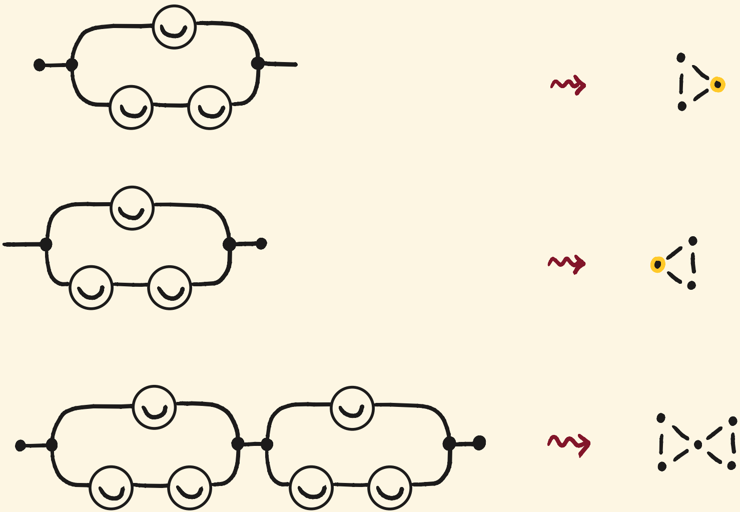
FROBENIUS



[Bonchi, Seeber & Sobociński 2018]

GRAPHS AS MORPHISMS - EXAMPLE

graphs with sources are τ-structures with $\tau =$ 



BRANCH WIDTH [Robertson & Seymour, 1991]

G undirected graph

BRANCH DECOMPOSITION

(Y, b) where

- Y is a subcubic tree (= any node has at most 3 neighbours)
- b : leaves$(Y) \xrightarrow{\simeq}$ edges(G) labelling bijection

WIDTH OF (Y, b)

$$\mathrm{wd}(Y,b) := \max_{e \in \mathrm{edges}\, Y} |\mathrm{ends}\, A_e \cap \mathrm{ends}\, B_e|$$

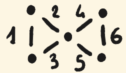
$\{A_e, B_e\}$ partition of E given by e through b

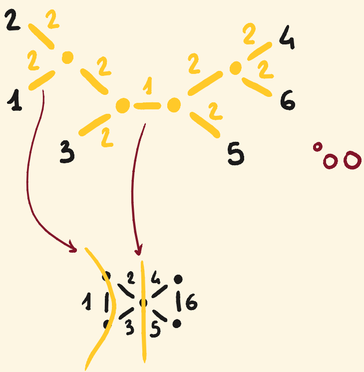
BRANCH WIDTH

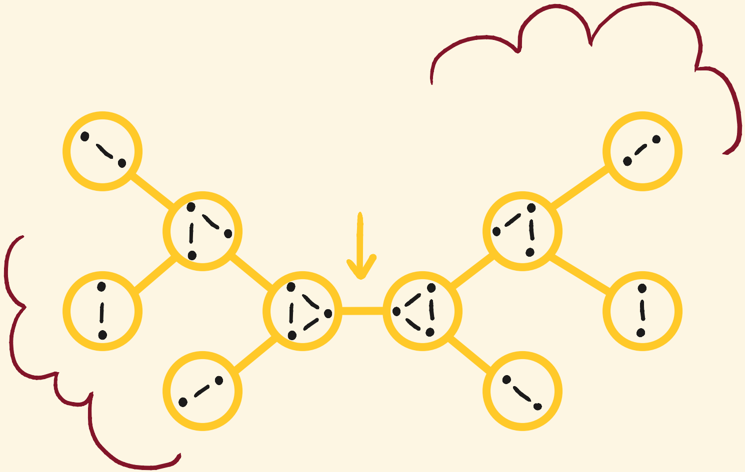
$$\mathrm{bwd}(G) := \min_{(Y,b)} \mathrm{wd}(Y,b)$$

⇝ cost of a cheapest decomposition

BRANCH WIDTH - EXAMPLE

$G =$ 





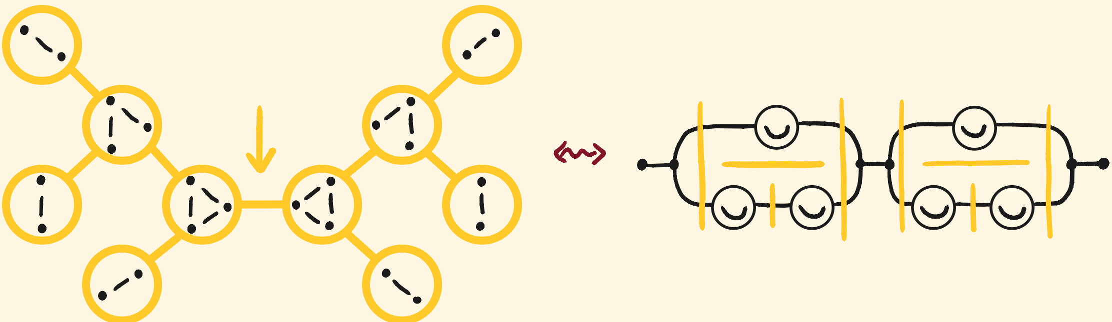
BRANCH WIDTH & MONOIDAL WIDTH

$G = (V, E)$ undirected graph

$g = \emptyset \xrightarrow{i} G \xleftarrow{i} \emptyset : \emptyset \to \emptyset$ in $\mathrm{cospan}(\mathrm{UGraph})_\emptyset$

THEOREM

$$\frac{1}{2}\,\mathrm{bwd}(G) \leq \mathrm{mwd}(g) \leq \mathrm{bwd}(G) + 1$$



[DL & Sobociński 2022]

OUTLINE

- Monoidal decompositions
- Matrices
- Rank width
- Branch width
- [Fixed-parameter tractability]

COMPOSITIONAL ALGORITHMS

$\mathcal{C}, \mathcal{D}$ monoidal categories

$P : \mathcal{C} \to \mathcal{D}$ monoidal functor

space of solutions

$w : \mathrm{morph}\, \mathcal{C} \to \mathbb{N}$ weight function

a <u>compositional algorithm</u> for P w.r.t. w computes

1. $P(f)$ in time $\mathcal{O}(c(w(f)) \cdot w(f))$
2. $P(f) ;_Y P(g)$ in $\mathcal{D}$ in time $\mathcal{O}(c(w(\mathbb{1}_Y)) \cdot (w(f) + w(g)))$
3. $P(f) \otimes P(f')$ in $\mathcal{D}$ in time $\mathcal{O}(c(1) \cdot (w(f) + w(f')))$

for some function $c : \mathbb{N} \to \mathbb{N}$.

usually more than exponential

⇝ cf. MSOL-smooth operations and MSOL-inductive classes of τ-structures

⇝ cf. number of vertices

[cf. Courcelle & Makowsky 2002]

FEFERMAN-VAUGHT THEOREM

THEOREM

For τ-structures A, B, A', B' and a set X of sources,
if $A \equiv_{MSO(\tau)} A'$ and $B \equiv_{MSO(\tau)} B'$, then $A \oplus_X B \equiv_{MSO(\tau)} A' \oplus_X B'$.
Computing $A \oplus_X B \models \varphi$ given $Th_{MSO_q(\tau)}(A)$ and $Th_{MSO_q(\tau)}(B)$ does not depend on $A \oplus_X B$.

COROLLARY

There is a monoidal functor

$$\begin{aligned} Th : Struct(\tau) &\longrightarrow Struct(\tau)/_{\equiv_{MSO(\tau)}} \\ A &\longmapsto Th_{MSO_q(\tau)}(A) \end{aligned}$$

that can be computed with a compositional algorithm.

[Feferman & Vaught 1959, Courcelle & Makowsky 2002]

MONOIDAL FIXED-PARAMETER TRACTABILITY

THEOREM

Computing a functorial problem $P : \mathcal{C} \to \mathcal{D}$ with a compositional algorithm w.r.t. $w : \mathrm{morph}\,\mathcal{C} \to \mathbb{N}$ is fixed-parameter tractable with parameter monoidal width.
Explicitly, if $\mathrm{mwd}(g) \leq k$, computing $P(g)$ takes $\mathcal{O}(c(k) \cdot w(g))$, for some $c : \mathbb{N} \to \mathbb{N}$.

COROLLARY

Checking an MSO formula $\varphi \in MSO_q$ on τ-structures is fixed-parameter tractable with parameter tree width.

SUMMARY & FUTURE DIRECTIONS

- Monoidal width measures structural complexity of morphisms in monoidal categories.
- Monoidal width captures rank width and tree width.
- We would like to find other examples of fixed-parameter tractability more in the spirit of morphisms as processes.