Cambridge 10th November 2022

MONOIDAL WIDTH

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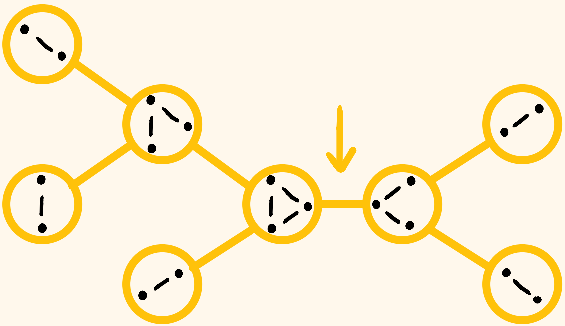
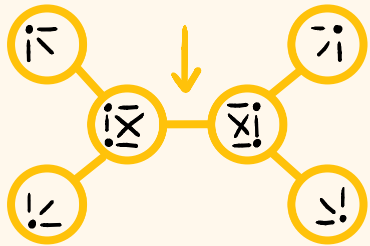
OVERVIEW

- graph parameters like tree width, branch width and rank width are technically different but intuitively similar

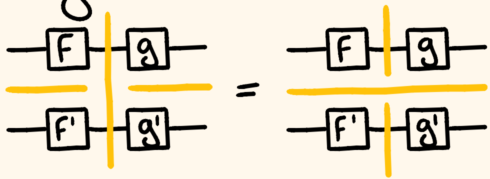




- are they instances of a more general 'width'?

RESULTS

- we define monoidal width to measure the difficulty of decomposing morphisms in monoidal categories



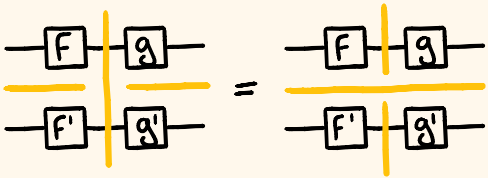
- we capture rank width, branch width, tree width and path width by instantiating monoidal width in suitable categories of graphs

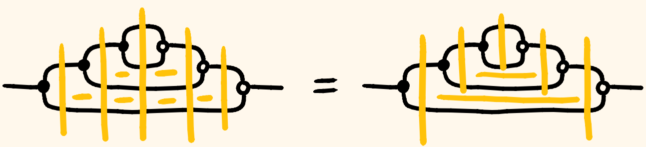
OUTLINE

- [monoidal decompositions]
- monoidal width for matrices
- monoidal width for rank width
- monoidal width for branch (tree, path) width

WIDTHS AND DECOMPOSITIONS

- each width is based on a notion of decomposition
- we decompose morphisms using the operations allowed in monoidal categories: compositions and monoidal products





DECOMPOSITIONS

In a monoidal category $\mathcal{C}$ $\rightsquigarrow$ FinSet

- $\mathcal{O} = \{\otimes, ;_X \text{ for } X \in \mathrm{obj}(\mathcal{C})\}$: set of operations
- $\mathcal{A}$: set of 'atomic' morphisms in $\mathcal{C}$

  $\rightsquigarrow \mathcal{A} = \{\supset\!-, \bullet\!-, \times, -\}$

- $w : \mathcal{A} \cup \mathcal{O} \to \mathbb{N}$: weight function

  such that $\begin{cases} w(\otimes) = 0 \\ w(;_{X \otimes Y}) = w(;_X) + w(;_Y) \end{cases}$

  $\rightsquigarrow$ $w(\supset\!-) = w(\times) = 2$
  $w(\bullet\!-) = w(-) = 1$
  $w(;_n) = n$

MONOIDAL DECOMPOSITION

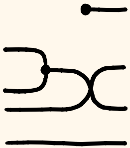
$f : X \to Y$ morphism in $\mathcal{C}$

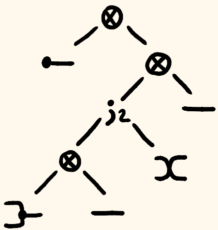
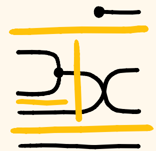
a monoidal decomposition $d \in \mathcal{D}_f$ of f is

$$
\begin{aligned}
d ::= \ & (f) && \text{if } f \in \mathcal{A} \\
\mid \ & d_1 \,{}^{\swarrow}\!{;_\mathcal{C}}\!{}^{\searrow}\, d_2 && \text{if } f = f_1 ;_\mathcal{C} f_2 \ , \ d_1 \in \mathcal{D}_{f_1} , \ d_2 \in \mathcal{D}_{f_2} \\
\mid \ & d_1 \,{}^{\swarrow}\!{\otimes}\!{}^{\searrow}\, d_2 && \text{if } f = f_1 \otimes f_2 \ , \ d_1 \in \mathcal{D}_{f_1} , \ d_2 \in \mathcal{D}_{f_2}
\end{aligned}
$$

$\rightsquigarrow$ it's a labelled binary tree

MONOIDAL DECOMPOSITION – EXAMPLE

 $: 4 \longrightarrow 4$ morphism in FinSet

$d =$  $\rightsquigarrow$ 

MONOIDAL WIDTH

$d \in \mathcal{D}_f$ monoidal decomposition of f

WIDTH OF d

$$\mathrm{wd}(d) := \max\{w(n) \mid n \in \mathrm{nodes}(d)\}$$

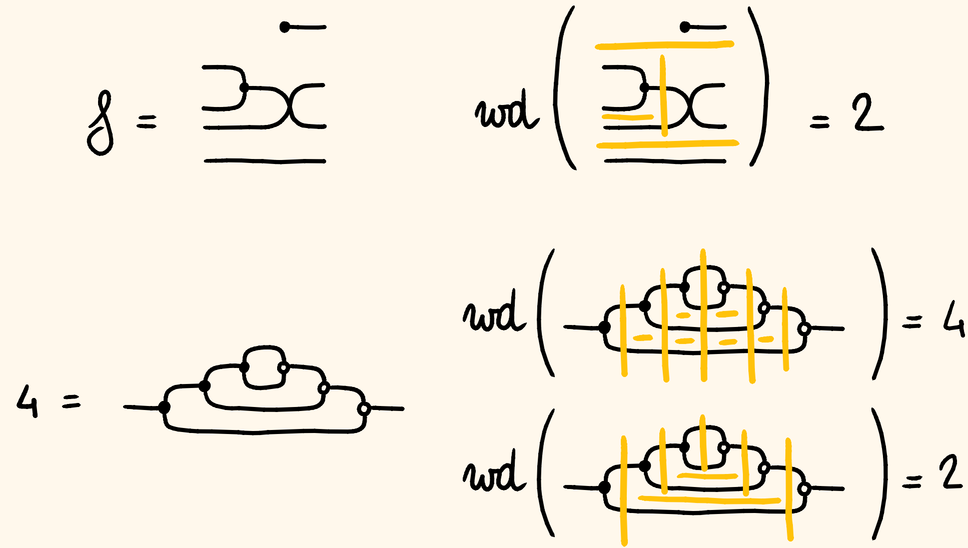
⇝ cost of the most expensive operation or atom

MONOIDAL WIDTH OF f

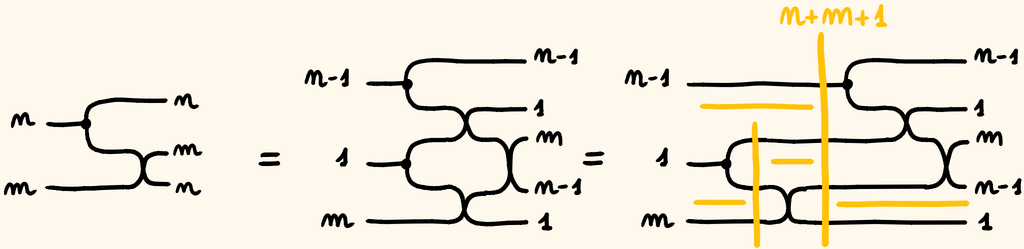
$$\mathrm{mwd}(f) := \min\{\mathrm{wd}(d) \mid d \in \mathcal{D}_f\}$$

⇝ cost of the cheapest decomposition

MONOIDAL WIDTH - EXAMPLE



MONOIDAL WIDTH OF COPYING



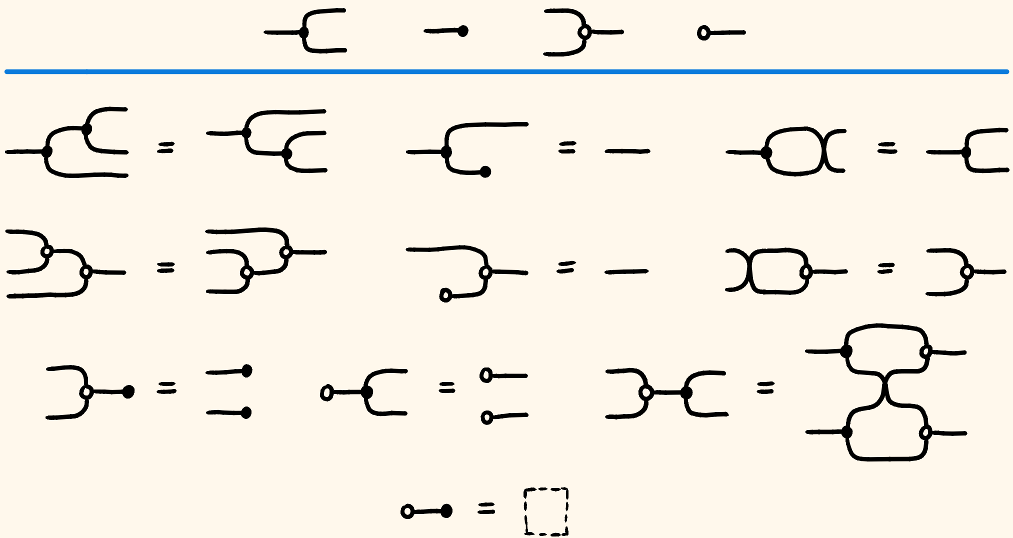
$m = 0$

$\Rightarrow \text{mwd}(\text{copy}_n) \leq n + 1$

OUTLINE

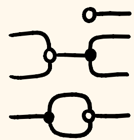
- monoidal decompositions
- [monoidal width for matrices]
- monoidal width for rank width
- monoidal width for branch (tree, path) width

PROP OF MATRICES



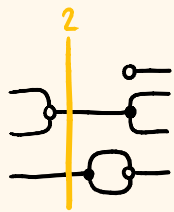
PROP OF MATRICES - EXAMPLE

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} =$$

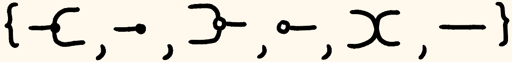


FACT : the minimal vertical cut in a matrix is its rank : $\min\{k \in \mathbb{N} \mid A = B ;_k C\} = \text{rank } A$

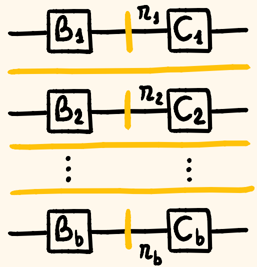
$\text{rank } A = 2$ $\rightsquigarrow$



MONOIDAL WIDTH OF MATRICES

$\mathcal{A} =$ 

$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & A_b \end{pmatrix} = A_1 \oplus A_2 \oplus \cdots \oplus A_b =$$

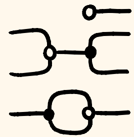


THEOREM

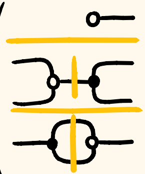
$$\max_i \operatorname{rank} A_i \leq \operatorname{mwd} A \leq \max_i \operatorname{rank} A_i + 1$$

MONOIDAL WIDTH OF MATRICES - EXAMPLE

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} =$$



$$\mathrm{wd}\left(\quad \right) = 2$$



$$= \max\{\mathrm{rank}(i),\ \mathrm{rank}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\ \mathrm{rank}(2)\} + 1$$

$$\mathrm{rank}(i) = 0, \quad \mathrm{rank}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 1, \quad \mathrm{rank}(2) = 1$$

OUTLINE

- monoidal decompositions
- monoidal width for matrices
- [monoidal width for rank width]
- monoidal width for branch (tree, path) width

OVERALL STRATEGY

1. we have a graph width
2. we find the relevant decomposition algebra to define an inductive version of the graph decomposition and find the appropriate monoidal category
3. bound monoidal width with the graph width by mapping monoidal decompositions to graph decompositions and viceversa

RANK WIDTH [Oum & Seymour, 2006]

$G = (V, E, \text{ends}: E \to \mathcal{P}_{\leq 2}(V))$ undirected graph

RANK DECOMPOSITION

(Y, r) where

- Y is a subcubic tree (= any node has at most 3 neighbours)
- r : leaves $Y \xrightarrow{\simeq} V$ labelling bijection

WIDTH OF (Y, r)

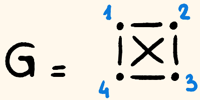
$$wd(Y, r) := \max_{e \in \text{edges}\, Y} \text{rank}(X_e)$$

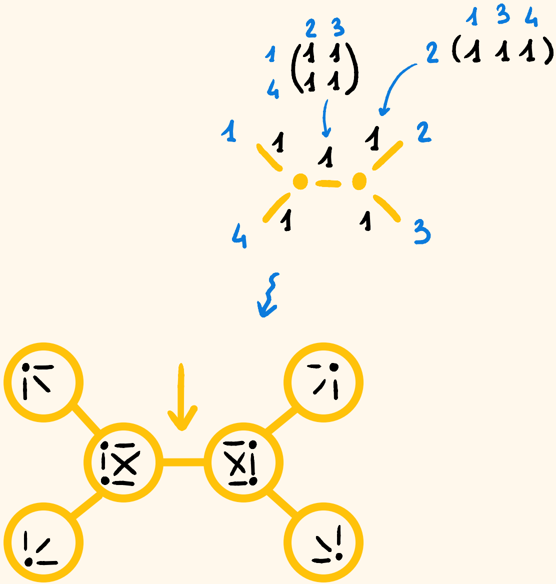
X_e adjacency matrix of the cut given by e through r

RANK WIDTH

$$rwd(G) := \min_{(Y, r)} wd(Y, r)$$

RANK WIDTH - EXAMPLE





GRAPHS WITH DANGLING EDGES

$G = (V, E)$ undirected graph ⇝ up to isomorphism

$\Rightarrow [G]$ with $G \in Mat_{\mathbb{N}}(k,k)$ and

$$[G] = [H] \iff G + G^T = H + H^T$$

$\Gamma = ([G], B)$ graph with dangling edges $B \in Mat_{\mathbb{N}}(k,n)$

$\Gamma =$  $= \left(\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right], \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right)$

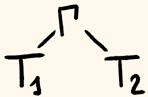
⇝ graphs can be 'glued' along their dangling edges

RANK DECOMPOSITIONS – RECURSIVELY

$\Gamma = ([G], B)$ graph with dangling edges

RECURSIVE RANK DECOMPOSITION

$T ::= (\Gamma)$ if Γ has at most one vertex

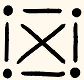
| 

if T_i rec. rank dec. of $\Gamma_i = ([G_i], B_i)$

$[G] = \left[\begin{pmatrix} G_1 & C \\ D & G_2 \end{pmatrix}\right]$, $B = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$

$B_1 = (A_1 | C)$; $B_2 = (A_2 | C^T)$

⇝ Γ is obtained by 'glueing' Γ_1 and Γ_2

 =  'glued' with 

RANK WIDTH – RECURSIVELY

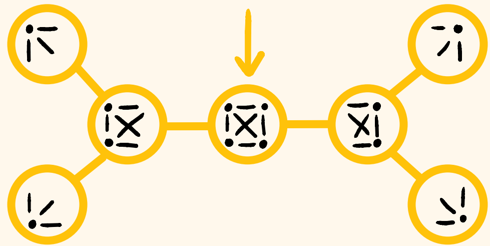
T recursive rank decomposition of $\Gamma = ([G], B)$

WIDTH OF T

$$wd(T) := \begin{cases} \operatorname{rank} B & \text{if } T = (\Gamma) \\ \max\{wd(T_1), \operatorname{rank} B, wd(T_2)\} & \text{if } T = \overset{\Gamma}{T_1 \quad T_2} \end{cases}$$

RECURSIVE RANK WIDTH

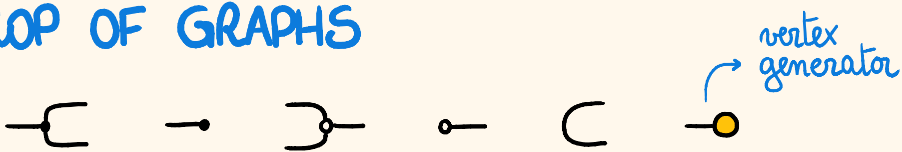
$$rrwd(\Gamma) := \min_T wd(T)$$



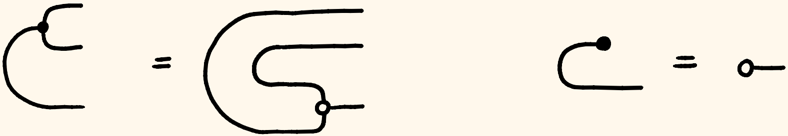
PROPOSITION

$$rwd(G) \leq rrwd(\Gamma) \leq rwd(G) + \operatorname{rank} B$$

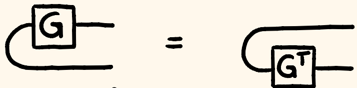
A PROP OF GRAPHS



bialgebra equations +

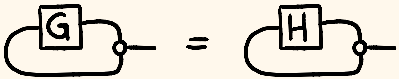


⇝ the cup transposes



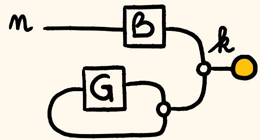
and captures equivalence of adjacency matrices

[G] = [H] ⟺

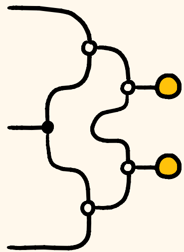


A PROP OF GRAPHS – EXAMPLE

$\Gamma = ([G], B) \rightsquigarrow$

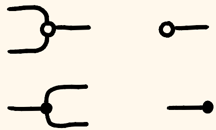


 $= \left(\left[\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right], \begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 1\end{pmatrix}\right) \rightsquigarrow$



DECOMPOSITIONS IN THE PROP OF GRAPHS

Bialgebra structure



+ 'vertex' generator

ATOMS

$\mathcal{A} = \{\text{all morphisms}\}$

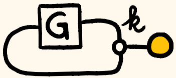
WEIGHT FUNCTION

$w(g) := |\text{vertices } g|$

$w(i_n) := n$

RANK WIDTH & MONOIDAL WIDTH

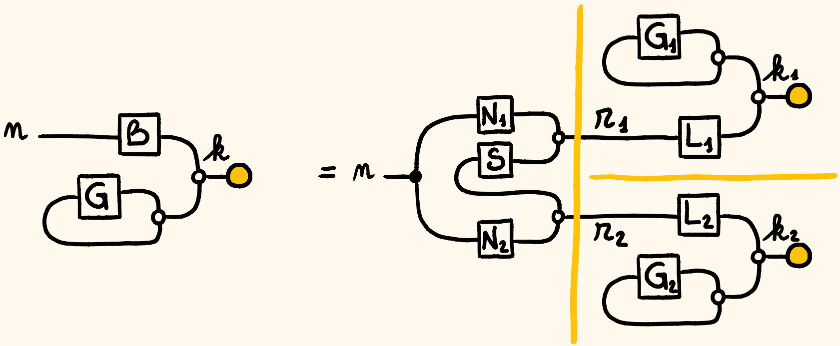
[G] undirected graph

$g =$ 

$: 0 \to 0$ in Graph

THEOREM

$$\frac{1}{2}\,\mathrm{rwd}(G) \leq \mathrm{mwd}(g) \leq 2\,\mathrm{rwd}(G)$$



OUTLINE

- monoidal decompositions
- monoidal width for matrices
- monoidal width for rank width
- [monoidal width for branch (tree, path) width]

BRANCH WIDTH [Robertson & Seymour, 1991]

$G = (V, E, \text{ends} : E \to \mathcal{P}_{\leq 2}(V))$ undirected graph

BRANCH DECOMPOSITION

(Y, b) where

- Y is a subcubic tree (= any node has at most 3 neighbours)
- b : leaves $Y \xrightarrow{\simeq} E$ labelling bijection

WIDTH OF (Y, b)

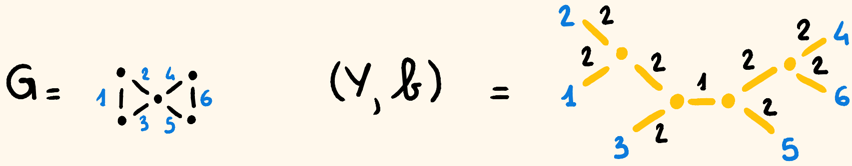
$$\text{wd}(Y, b) := \max_{e \in \text{edges } Y} |\text{ends } A_e \cap \text{ends } B_e|$$

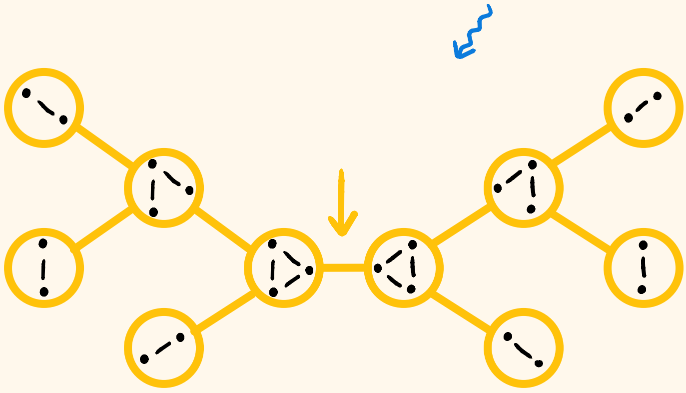
$\{A_e, B_e\}$ partition of E given by e through b

BRANCH WIDTH

$$\text{bwd}(G) := \min_{(Y, b)} \text{wd}(Y, b)$$

BRANCH WIDTH – EXAMPLE

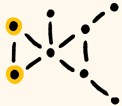




GRAPHS WITH SOURCES

$G = (V, E)$ undirected graph

$\Gamma = (G, X)$ graph with sources $X \subseteq V$

$\Gamma =$ 

⇝ graphs can be 'glued' along their sources

BRANCH DECOMPOSITIONS – RECURSIVELY

$\Gamma = ((V, E), X)$ graph with sources

RECURSIVE BRANCH DECOMPOSITION

$T ::= ()$ if Γ is discrete

$\mid (\Gamma)$ if Γ has one edge

$\mid \Gamma(T_1, T_2)$ — a node Γ with children T_1 and T_2 — if T_i rec. branch dec. of $\Gamma_i = ((V_i, E_i), X_i)$

$V = V_1 \cup V_2$, $E = E_1 \sqcup E_2$

$X_i = (V_1 \cap V_2) \cup (V_i \cap X)$

⇝ Γ is obtained by 'glueing' Γ_1 and Γ_2

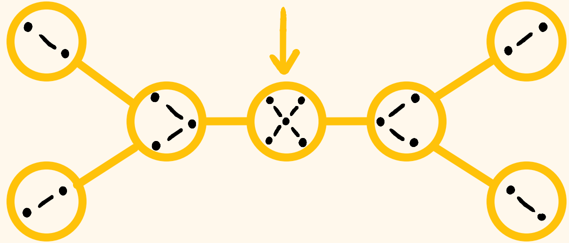
 =  'glued' with 

BRANCH WIDTH – RECURSIVELY

T recursive branch decomposition of $\Gamma = (G, X)$

WIDTH OF T

$$wd(T) := \begin{cases} 0 & \text{if } T = () \\ \max\{wd(T_1), |X|, wd(T_2)\} & \text{if } T = \begin{smallmatrix} & \Gamma & \\ T_1 & & T_2 \end{smallmatrix} \end{cases}$$



RECURSIVE BRANCH WIDTH

$$rbwd(\Gamma) := \min_T wd(T)$$

PROPOSITION

$$bwd(G) \leq rbwd(\Gamma) \leq bwd(G) + |X|$$

COSPANS OF GRAPHS

$\mathbf{Cospan}(\mathbf{UGraph})_\emptyset$

objects : sets ⇝ discrete graphs

morphisms $X \to Y$: cospans $X \xrightarrow{\partial_X} G \xleftarrow{\partial_Y} Y$ of graphs

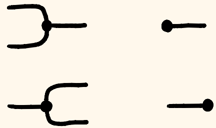
composition : by pushout ⇝ glue along vertices

monoidal product : component-wise disjoint union

⇝ graphs with left and right sources

DECOMPOSITIONS IN COSPANS OF GRAPHS

Frobenius structure



+ 'edge' generator

ATOMS

$\mathcal{A} = \{ \text{all morphisms} \}$

WEIGHT FUNCTION

$$w\left(X \rightarrow (V,E) \leftarrow Y \right) := |V|$$

$$w\left(j_X \right) := |X|$$

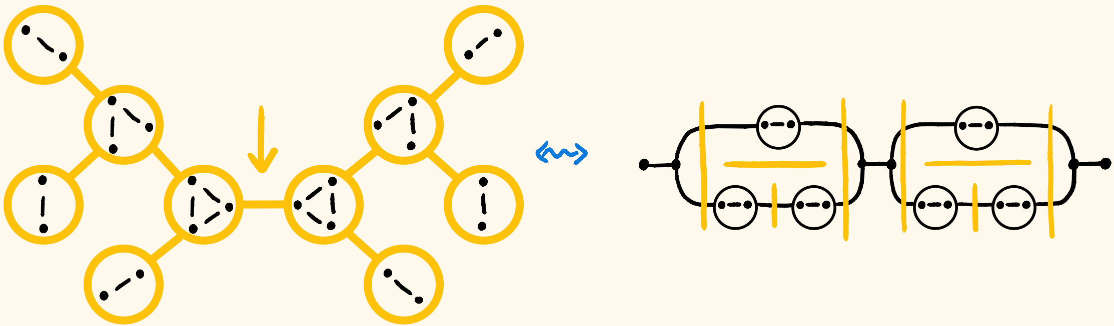
BRANCH WIDTH & MONOIDAL WIDTH

$G = (V, E)$ undirected graph

$g = \varnothing \xrightarrow{i} G \xleftarrow{i} \varnothing : \varnothing \to \varnothing$ in $\mathrm{Cospan}(\mathrm{UGraph})_\varnothing$

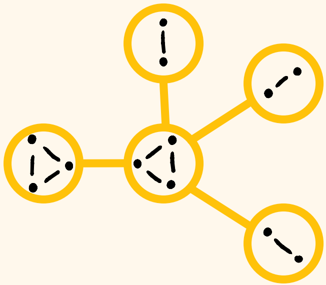
THEOREM

$$\frac{1}{2}\,\mathrm{bwd}(G) \leq \mathrm{mwd}(g) \leq \mathrm{bwd}(G) + 1$$



PATH AND TREE WIDTHS

- path and tree decompositions rely on the same algebra as branch width but restrict the 'shape' of decompositions



- we restrict the allowed operations

PATH WIDTH [Robertson & Seymour, 1983]

$G = (V, E, \mathrm{ends} : E \to \mathcal{P}_{\leq 2}(V))$ undirected graph

PATH DECOMPOSITION

(P, p) where

- $P = \overset{1}{\bullet} - \overset{2}{\bullet} - \dots \overset{n}{\bullet}$ is a path
- $p : \mathrm{vertices}\, P \longrightarrow \mathcal{P}(V)$ labelling function

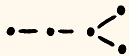
such that

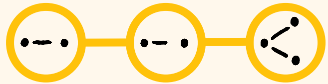
- $\bigcup \{ p(i) : i \in \mathrm{vertices}\, P \} = V$
- $\forall e \in E \;\; \exists i \in \mathrm{vertices}\, P \;\; \mathrm{ends}(e) \subseteq p(i)$

  ⇝ cover all the graph (both conditions above)

- $\forall i < j < k \in \mathrm{vertices}\, P \;\; p(i) \cap p(k) \subseteq p(j)$ ⇝ path shape

PATH WIDTH – EXAMPLE

$G =$ 

$(P,p) =$ 

WIDTH OF (P,p)

Robertson & Seymour

$$wd(P,p) := \max_{i \in \text{vertices } P} |p(i)| \quad (-1) \quad \rightsquigarrow \quad wd(P,p) = 3$$

PATH WIDTH

$$pwd(G) := \min_{(P,p)} wd(P,p) \quad \rightsquigarrow \quad pwd(G) = 3$$

PATH DECOMPOSITIONS – RECURSIVELY

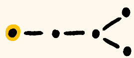
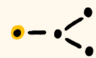
$\Gamma = ((V,E),X)$ graph with sources

RECURSIVE PATH DECOMPOSITION

$T ::= ()$ if $\Gamma = \emptyset$

$\quad\mid (V_1, T')$ if T' rec. path dec. of $\Gamma' = ((V',E'),X')$
$V = V_1 \cup V'$, $X \subseteq V_1$
$X' = V_1 \cap V'$, $\mathrm{ends}(E \setminus E') \subseteq V_1$

⇝ Γ is obtained by 'glueing' Γ' with $\Gamma_1 := ((V_1, E \setminus E'), X \cup X')$

 =  'glued' with 

PATH WIDTH - RECURSIVELY

T recursive path decomposition of $\Gamma = (G, X)$

WIDTH OF T

$$wd(T) := \begin{cases} 0 & \text{if } T = () \\ \max\{|V_1|, wd(T')\} & \text{if } T = (V_1, T') \end{cases}$$

RECURSIVE PATH WIDTH

$$rpwd(\Gamma) := \min_T wd(T)$$

PROPOSITION

$$rpwd(\Gamma) = pwd(G)$$

MONOIDAL PATH WIDTH

$\rightsquigarrow$ ban monoidal products completely

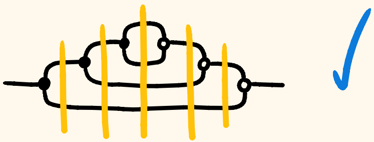
$f : X \to Y$ in $\mathcal{C}$

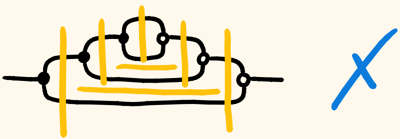
a monoidal path decomposition $d \in \mathcal{D}^{p}_{f}$ of f is

$$d ::= (f) \qquad \text{if } f \in \mathcal{A}$$

$$\mid\ d_1 \overset{;_C}{\frown} d_2 \qquad \text{if } f = f_1 ;_C f_2 ,\ d_1 \in \mathcal{D}^{p}_{f_1} ,\ d_2 \in \mathcal{D}^{p}_{f_2}$$

MONOIDAL PATH WIDTH - EXAMPLE





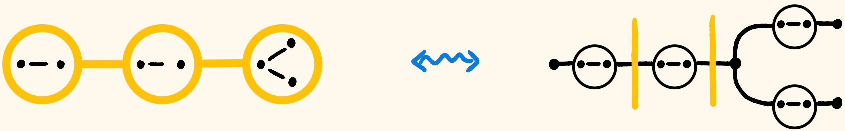
PATH WIDTH & MONOIDAL WIDTH

$G = (V, E)$ undirected graph

$g = \emptyset \xrightarrow{i} G \xleftarrow{i} \emptyset : \emptyset \to \emptyset$ in $\mathsf{cospan}(\mathsf{UGraph})_\emptyset$

THEOREM

$\mathsf{pwd}(G) = \mathsf{mpwd}(g)$



TREE WIDTH [Robertson & Seymour, 1986]

$G = (V, E, \text{ends} : E \to \mathcal{P}_{\leq 2}(V))$ undirected graph

TREE DECOMPOSITION

(Y, t) where

- Y is a tree (= connected acyclig graph)
- $t : \text{vertices}\ Y \longrightarrow \mathcal{P}(V)$ labelling function

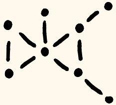
such that

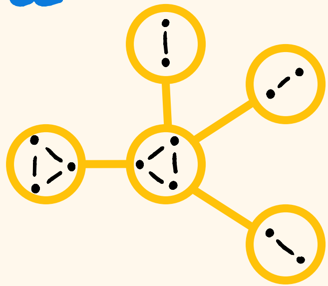
- $\bigcup \{ t(i) : i \in \text{vertices}\ Y \} = V$
- $\forall e \in E\ \exists i \in \text{vertices}\ Y\ \ \text{ends}(e) \subseteq t(i)$

⇝ cover all the graph

- $\forall\ i \rightsquigarrow j \rightsquigarrow k \in \text{vertices}\ Y\ \ t(i) \cap t(k) \subseteq t(j)$ ⇝ tree shape

TREE WIDTH – EXAMPLE

$G =$ 

$(Y, t) =$ 

WIDTH OF (Y, t)

Robertson & Seymour

$$wd(Y,t) := \max_{i \in \text{vertices } Y} |t(i)| \quad (-1) \quad \rightsquigarrow \quad wd(Y,t) = 3$$

TREE WIDTH

$$twd(G) := \min_{(Y,t)} wd(Y,t) \quad \rightsquigarrow \quad twd(G) = 3$$

TREE DECOMPOSITIONS – RECURSIVELY

$\Gamma = ((V, E), X)$ graph with sources

RECURSIVE TREE DECOMPOSITION

$T ::= ()$ if $\Gamma = \emptyset$

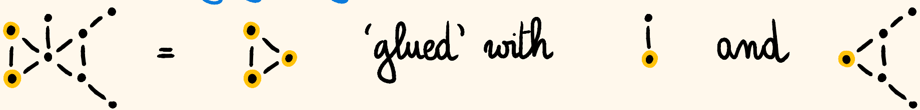
$\mid$ V' with subtrees T_1 and T_2 if T_i rec. tree dec. of $\Gamma_i = ((V_i, E_i), X_i)$

$V = V_1 \cup V' \cup V_2$, $X \subseteq V'$

$X_i = V_i \cap V'$, $V_1 \cap V_2 \subseteq V'$

$E_1 \cap E_2 = \emptyset$, $\mathrm{ends}(E \setminus (E_1 \sqcup E_2)) \subseteq V'$

⇝ Γ is obtained by 'glueing' Γ_1 and Γ_2 with $\Gamma' := ((V', E \setminus (E_1 \sqcup E_2)), X \cup X_1 \cup X_2)$



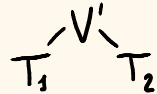
TREE WIDTH – RECURSIVELY

T recursive tree decomposition of $\Gamma = (G, X)$

WIDTH OF T

$$wd(T) := 0 \qquad \text{if } T = ()$$

$$\mid \max\{wd(T_1), |V'|, wd(T_2)\} \qquad \text{if } T =$$



RECURSIVE TREE WIDTH

$$rtwd(\Gamma) := \min_T wd(T)$$

PROPOSITION

$$rtwd(\Gamma) = twd(G)$$

MONOIDAL TREE DECOMPOSITION

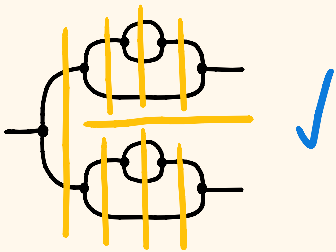
⇝ restrict compositions to have an atom on one side and recursion on the other

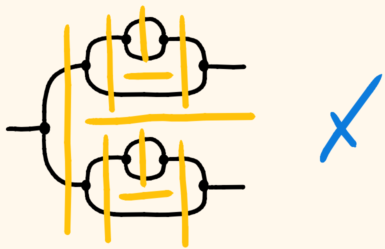
$f : X \to Y$ in $\mathcal{C}$

a monoidal tree decomposition $d \in \mathcal{D}^T_f$ of f is

$$
\begin{aligned}
d ::= \; & (f) && \text{if } f \in \mathcal{A} \\
\mid \; & (g) \overset{;_{\mathcal{C}}}{\frown} d' && \text{if } f = g \,;_{\mathcal{C}} f' \,,\; g \in \mathcal{A} \,,\; d' \in \mathcal{D}^T_{f'} \\
\mid \; & d_1 \overset{\otimes}{\frown} d_2 && \text{if } f = f_1 \otimes f_2 \,,\; d_1 \in \mathcal{D}^T_{f_1} \,,\; d_2 \in \mathcal{D}^T_{f_2}
\end{aligned}
$$

MONOIDAL TREE DECOMPOSITION – EXAMPLE





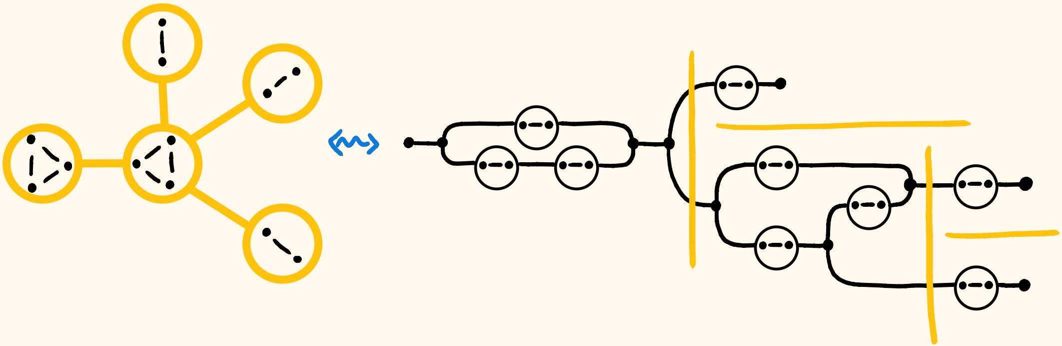
TREE WIDTH & MONOIDAL WIDTH

$G = (V, E)$ undirected graph

$g = \emptyset \xrightarrow{i} G \xleftarrow{i} \emptyset : \emptyset \to \emptyset$ in $\mathrm{Cospan}(\mathrm{UGraph})_{\emptyset}$

THEOREM

$$\mathrm{twd}(G) \leq \mathrm{mtwd}(g) \leq 2 \cdot \mathrm{twd}(G)$$



SUMMARY OF RESULTS

MATRICES

$$\max_i \operatorname{rank} A_i \leq \operatorname{mwd} A \leq \max_i \operatorname{rank} A_i + 1$$

COSPANS OF GRAPHS

$$\operatorname{pwd}(G) = \operatorname{mpwd}(g)$$

$$\operatorname{twd}(G) \leq \operatorname{mtwd}(g) \leq 2 \cdot \operatorname{twd}(G)$$

$$\frac{1}{2} \operatorname{bwd}(G) \leq \operatorname{mwd}(g) \leq \operatorname{bwd}(G) + 1$$

PROP OF GRAPHS

$$\frac{1}{2} \operatorname{rwd}(G) \leq \operatorname{mwd}(g) \leq 2 \operatorname{rwd}(G)$$

FUTURE WORK

- monoidal width in other interesting categories ?
- study properties of monoidal width ?
  (tree width - branch width bounds,
  invariance under 'bisimulation', ...')
- how to compute optimal monoidal decompositions ?
- Courcelle's - like theorem for monoidal width
  or other algorithmic properties ?

THANKS FOR LISTENING 

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THIS WORK

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