ACT 2022 22 July 2022

MONOIDAL WIDTH

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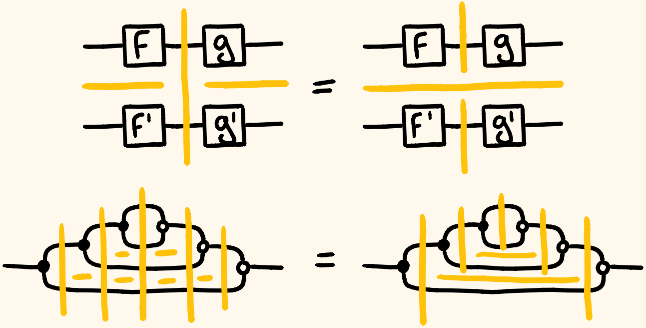
MOTIVATION (1)

- how efficient is to compute the semantics of morphisms in monoidal categories?



- we need an 'algebra of decompositions'

MOTIVATION (2)

- existing notions of complexity for graphs are based on decompositions: path width, tree width, branch width and rank width
- make explicit the algebra of decomposition that is hidden behind the definitions of these graph widths

MAIN RESULTS

- monoidal width as a measure of complexity for morphisms in monoidal categories
- monoidal decomposition as explicit decomposition algebra
- capture some known measures of complexity for graphs: path width, tree width, branch width and rank width

OUTLINE

- [monoidal decompositions]
- monoidal width for matrices
- monoidal width for rank width

DECOMPOSITION SYSTEM

A decomposition system $(\mathcal{A}, \mathcal{O}, w)$ in a monoidal category $\mathcal{C}$ is given by

- $\mathcal{A}$: set of 'atomic' morphisms in $\mathcal{C}$
- $\mathcal{O} = \{\otimes, ;_X \text{ for } X \in obj(\mathcal{C})\}$: set of operations
- $w : \mathcal{A} \cup \mathcal{O} \to \mathbb{N}$: weight function

  such that:

$$\begin{cases} w(\otimes) = 0 \\ w(;_{X \otimes Y}) = w(;_X) + w(;_Y) \end{cases}$$

DECOMPOSITION SYSTEM - EXAMPLE

a decomposition system $(\mathcal{A}, \mathcal{O}, w)$

in $\mathcal{C}$ $\rightsquigarrow$ $\mathcal{F}in\mathcal{S}et$

- $\mathcal{A}$: set of 'atoms' $\rightsquigarrow$ $\{ \supset\!-, \bullet\!-, \times, - \}$

- $\mathcal{O} = \{ \otimes , ;_X \text{ for } X \in obj(\mathcal{C}) \}$: set of operations

- $w : \mathcal{A} \cup \mathcal{O} \longrightarrow \mathbb{N}$: weight
  such that:

$$\begin{cases} w(\otimes) = 0 \\ w(;_{X \otimes Y}) = w(;_X) + w(;_Y) \end{cases}$$

$\rightsquigarrow$ $w(\supset\!-) = w(\times) = 2$

$w(\bullet\!-) = w(-) = 1$

$w(;_n) = n$

MONOIDAL DECOMPOSITION

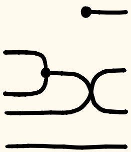
$f : X \to Y$ morphism in $\mathcal{C}$

a monoidal decomposition $d \in \mathcal{D}_f$ of f is

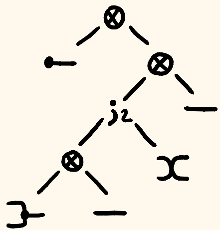
$$
\begin{aligned}
d ::= \; & (f) && \text{if } f \in \mathcal{A} \\
\mid \; & d_1 \overset{;_\mathcal{C}}{\frown} d_2 && \text{if } f = f_1 ;_\mathcal{C} f_2 ,\; d_1 \in \mathcal{D}_{f_1} ,\; d_2 \in \mathcal{D}_{f_2} \\
\mid \; & d_1 \overset{\otimes}{\frown} d_2 && \text{if } f = f_1 \otimes f_2 ,\; d_1 \in \mathcal{D}_{f_1} ,\; d_2 \in \mathcal{D}_{f_2}
\end{aligned}
$$

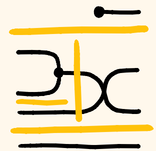
$\rightsquigarrow$ it's a labelled binary tree

MONOIDAL DECOMPOSITION - EXAMPLE



$: 4 \longrightarrow 4$ morphism in $\mathcal{F}in\mathcal{S}et$

$d =$ 

$\rightsquigarrow$ 

MONOIDAL WIDTH

$d \in \mathcal{D}_g$ monoidal decomposition of g

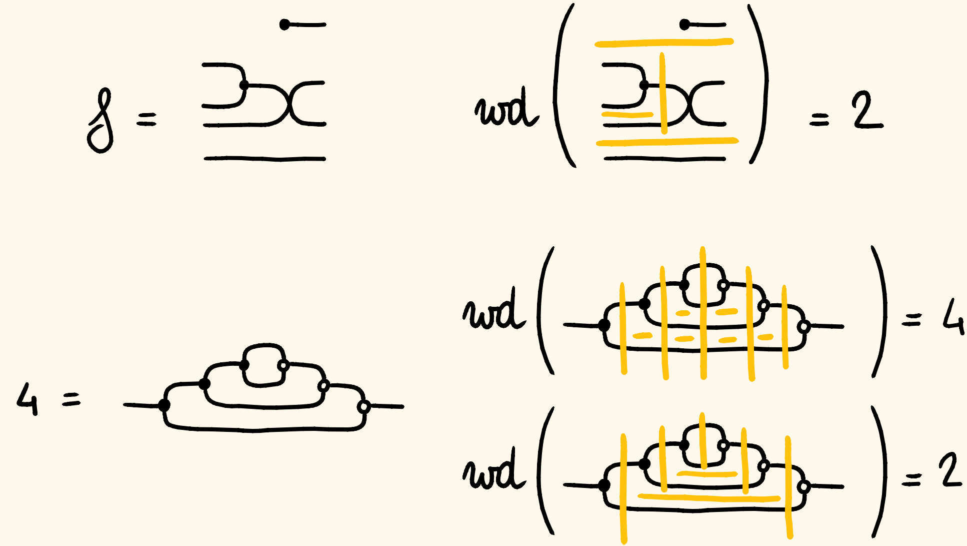
WIDTH OF d

$$
\mathrm{wd}(d) := \begin{cases} w(g) & \text{if } d = (g) \\ \max\{\mathrm{wd}(d_1),\, w(;_c),\, \mathrm{wd}(d_2)\} & \text{if } d = d_1 \swarrow^{;_c} \searrow d_2 \\ \max\{\mathrm{wd}(d_1),\, \mathrm{wd}(d_2)\} & \text{if } d = d_1 \swarrow^{\otimes} \searrow d_2 \end{cases}
$$

MONOIDAL WIDTH OF g

$$
\mathrm{mwd}(g) := \min_{d \in \mathcal{D}_g} \mathrm{wd}(d)
$$

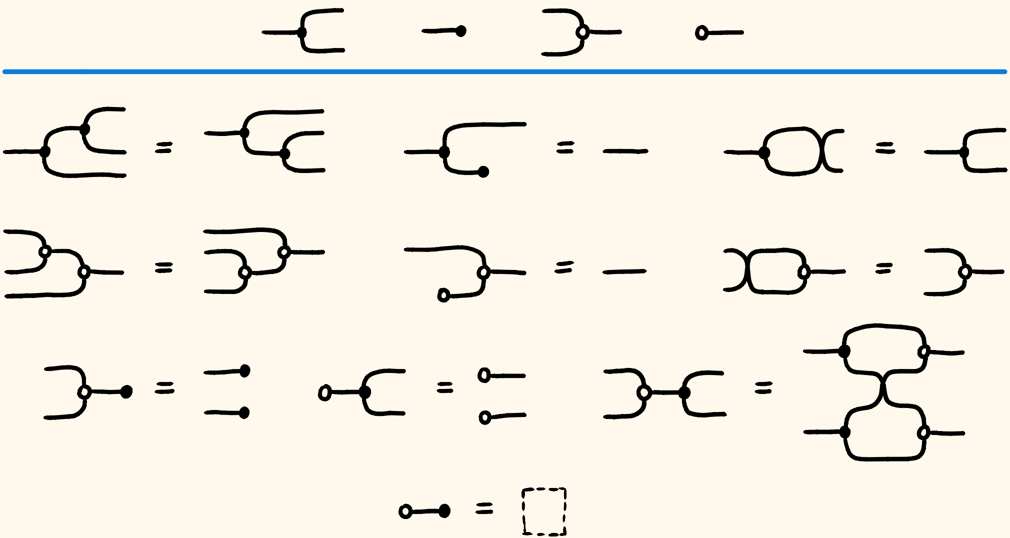
MONOIDAL WIDTH - EXAMPLE



OUTLINE

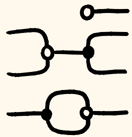
- monoidal decompositions
- [monoidal width for matrices]
- monoidal width for rank width

PROP OF MATRICES

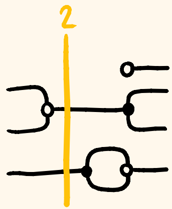


[1] Zanasi, Interacting Hopf algebras, PhD thesis (2018)

PROP OF MATRICES - EXAMPLE

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} =$$ 

FACT : the minimal vertical cut in a matrix is its rank : $\min\{k \in \mathbb{N} \mid A = B ;_k C\} = \text{rank } A$

rank $A = 2$ ⇝ 

MONOIDAL WIDTH OF MATRICES

$\mathcal{A} = \{$  $\}$

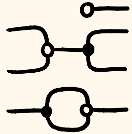
$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & A_b \end{pmatrix} = A_1 \oplus A_2 \oplus \cdots \oplus A_b$$

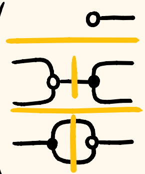
THEOREM

$$\max_i \operatorname{rank} A_i \leq \operatorname{mwd} A \leq \max_i \operatorname{rank} A_i + 1$$

MONOIDAL WIDTH OF MATRICES - EXAMPLE

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} =$$



$\text{wd}\Big($  $\Big) = 2$

$$= \max\Big\{\underbrace{\text{rank}(\,|\,)}_{=0},\ \underbrace{\text{rank}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}_{=1},\ \underbrace{\text{rank}(2)}_{=1}\Big\} + 1$$

OUTLINE

- monoidal decompositions
- monoidal width for matrices
- [monoidal width for rank width]

RANK WIDTH [Oum & Seymour, 2006]

$G = (V, E, \text{ends} : E \to \mathcal{P}_{\leq 2}(V))$ undirected graph

RANK DECOMPOSITION

(Y, r) where

- Y is a subcubic tree (= any node has at most 3 neighbours)
- r : leaves $Y \xrightarrow{\simeq} V$ labelling bijection

WIDTH OF (Y, r)

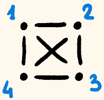
$$wd(Y, r) := \max_{e \in \text{edges} Y} \text{rank}(X_e)$$

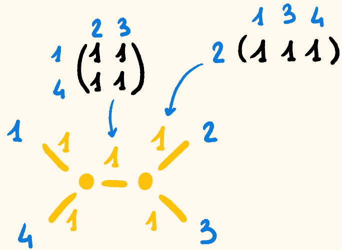
→ X_e adjacency matrix of the cut given by e through r

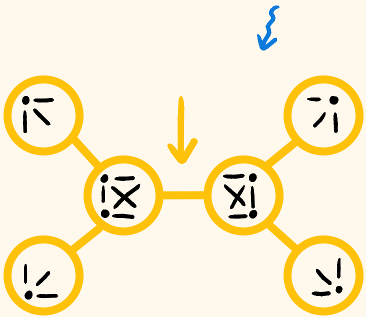
RANK WIDTH

$$rwd(G) := \min_{(Y, r)} wd(Y, r)$$

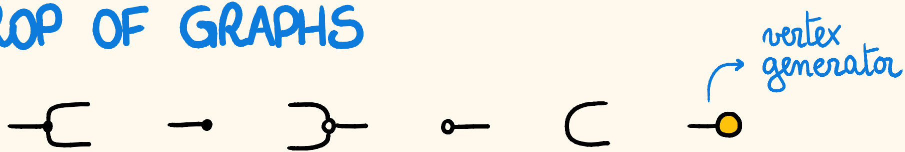
RANK WIDTH - EXAMPLE

G = 

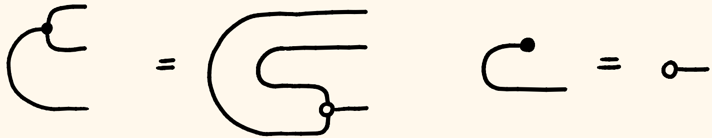




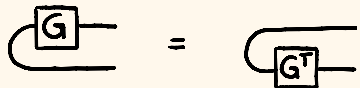
A PROP OF GRAPHS



bialgebra +

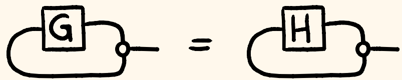


⇝ the cup transposes



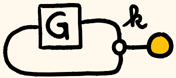
and captures equivalence of adjacency matrices

$G + G^T = H + H^T$ ⟺

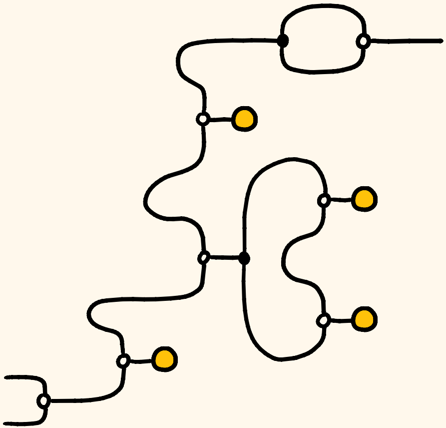


[3] Chantawibul & Sobociński, Towards compositional graph theory (2015)

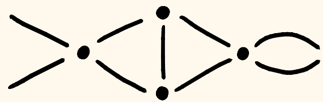
A PROP OF GRAPHS – EXAMPLE



⇝ graph on k vertices given by the adjacency matrix [G]

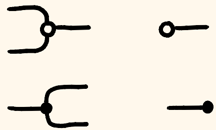


⇝



DECOMPOSITIONS IN THE PROP OF GRAPHS

Bialgebra structure



+ 'vertex' generator

ATOMS

$\mathcal{A} = \{\text{all morphisms}\}$

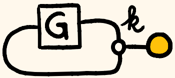
WEIGHT FUNCTION

$w(g) := |\text{vertices } g|$

$w(i_n) := n$

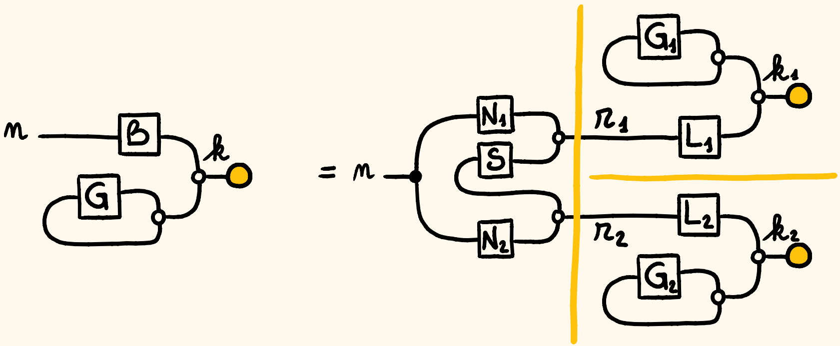
RANK WIDTH & MONOIDAL WIDTH

[G] undirected graph

$g =$  $: 0 \to 0$ in Grph

THEOREM

$$\frac{1}{2}\, \mathrm{rwd}(G) \leq \mathrm{mwd}(g) \leq 2\, \mathrm{rwd}(G)$$



SUMMARY OF RESULTS

MATRICES

$$\max_i \operatorname{rank} A_i \leq \operatorname{mwd} A \leq \max_i \operatorname{rank} A_i + 1$$

COSPANS OF GRAPHS

$$\operatorname{pwd}(G) = \operatorname{mpwd}(g)$$

$$\operatorname{twd}(G) \leq \operatorname{mtwd}(g) \leq 2 \cdot \operatorname{twd}(G)$$

$$\frac{1}{2} \operatorname{bwd}(G) \leq \operatorname{mwd}(g) \leq \operatorname{bwd}(G) + 1$$

PROP OF GRAPHS

$$\frac{1}{2} \operatorname{rwd}(G) \leq \operatorname{mwd}(g) \leq 2 \operatorname{rwd}(G)$$

FUTURE WORK

- obtain a result similar to Courcelle's theorem
- capture other widths (clique width, twin width, ...
  tree width for directed graphs and relational structures)
- algorithmic applications

SOME REFERENCES

[1] Zanasi, Interacting Hopf algebras, PhD thesis (2018)

[2] Oum & Seymour, Approximating clique width and branch width (2006)

[3] Chantawibul & Sobociński, Towards compositional graph theory (2015)

- Bonchi, Piedeleu, Sobociński & Zanasi, Graphical affine algebra (2019)
- Di Lavore, Hedges & Sobociński, Compositional modelling of network games (2021)

THIS WORK

- Di Lavore & Sobociński, Monoidal width: capturing rank width (2022)
- Di Lavore & Sobociński, Monoidal width: unifying tree width, path width and branch width (2022)