COMPOSITIONAL MODELLING OF NETWORK GAMES

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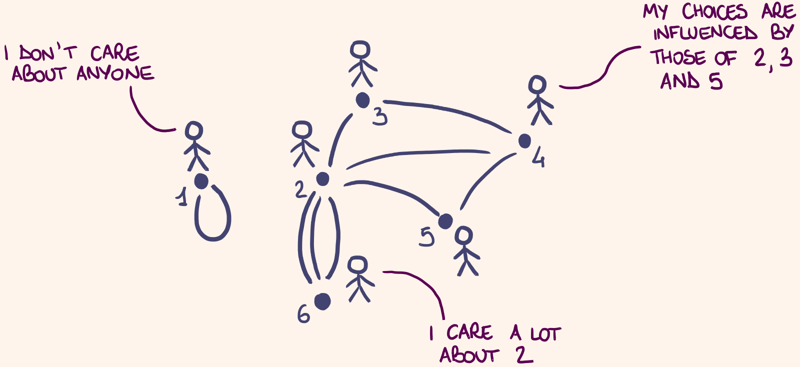
[Di Lavore, Hedges, Sobociński, Compositional modelling of network games, 2020]

OUTLINE

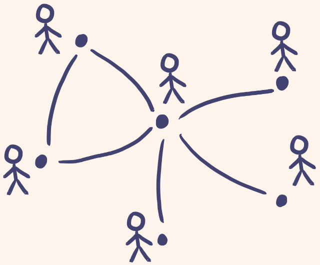
- $\boxed{\text{Games on graphs}}$
- The prop of open graphs
- Games on graphs as functors

GAMES ON GRAPHS



THE TRAGEDY OF THE COMMONS

- None of my neighbours invests ⇒ utility $1 - c + \varepsilon$
- One of my neighbours invests ⇒ utility 1
- I invest ⇒ utility $1 - c$



MAIN RESULT

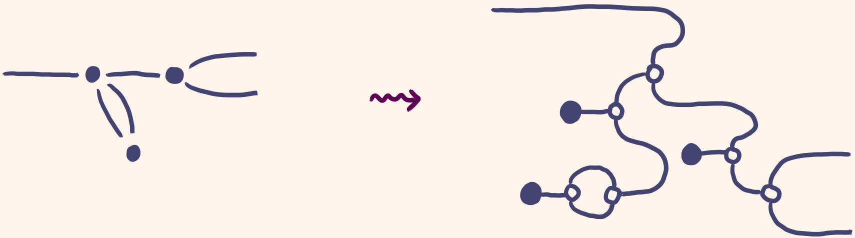
- Network games (a class of them) correspond to functors

$$\mathcal{G}raph \rightarrow \mathcal{G}ame$$

syntax $\longrightarrow$ semantics

MAIN RESULT

- Characterise the category of graphs with boundaries as the free category on some generators and equations.



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PROPS & STRING DIAGRAMS

 $: 2 \to 1$

 $: 1 \to 0$

sequential composition  $: 2 \to 0$

parallel composition  $: 3 \to 1$

FREE PROPS & FINSET

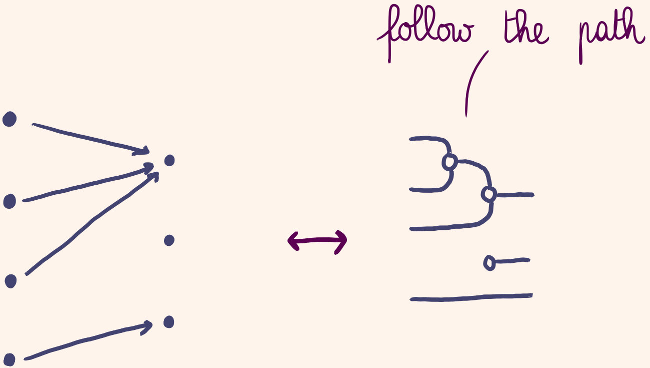
- FinSet is the free prop generated by

(associativity)

(unitality)

(commutativity)

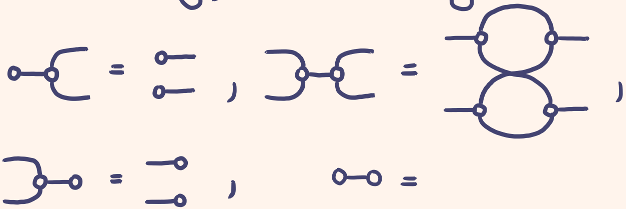
EXAMPLE IN FINSET



PROP OF MATRICES

- Natural numbers matrices are the free prop generated by

(co)associativity, (co) unitality, (co) commutativity,



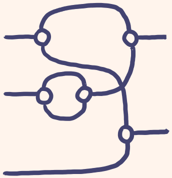
(bialgebra laws)

[S. Lack, Composing props, 2004]
[Bonchi, Sobociński, Zanasi, Interacting Hopf algebras, 2014]

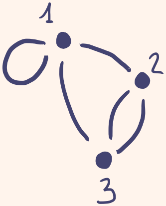
EXAMPLE IN $\mathbb{N}$-MATRICES

count the paths between inputs and outputs

$$\begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \leftrightarrow$$



GRAPHS & ADJACENCY MATRICES

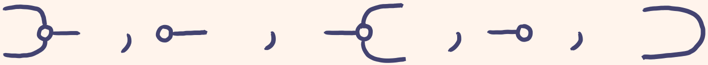


$$\longleftrightarrow \quad \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \end{array} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

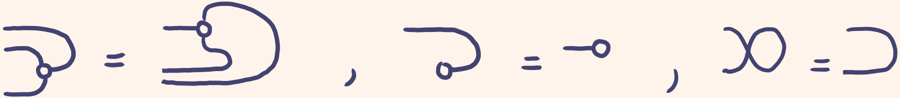
$$A \sim B \iff A + A^T = B + B^T$$

ADDING THE CUP

- Adjacency matrices are the $n \to 0$ morphisms in the free prop generated by



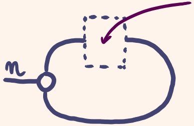
(co)associativity, (co)unitality, (co)commutativity, bialgebra laws



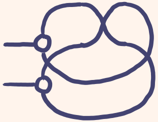
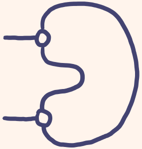
(cup laws)

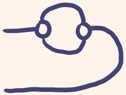
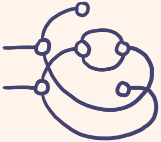
[Chantawibul, Sobociński, Towards compositional graph theory, 2015]

EXAMPLE ADJACENCY MATRICES



add here an $n \times n$ matrix

$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \longleftrightarrow$  = 

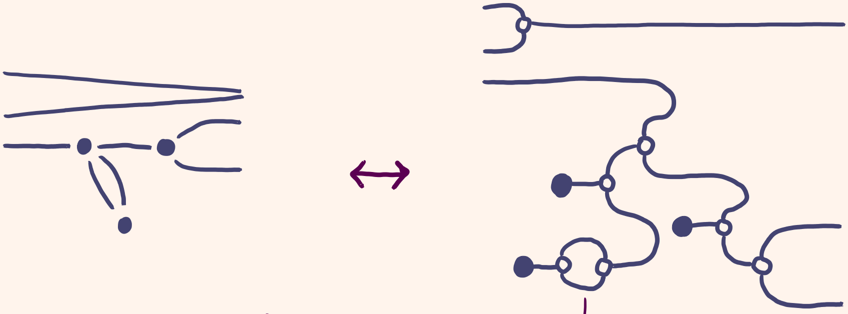
=  =  $\longleftrightarrow \begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix}$

ADDING VERTICES

- Graphs with boundaries are the free prop generated by

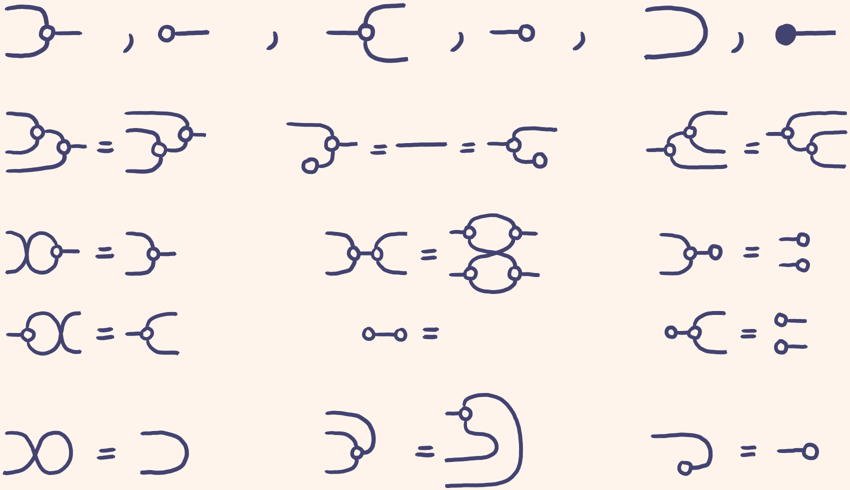
(co)associativity, (co)unitality,
bialgebra laws, cup laws

EXAMPLE OF GRAPH WITH BOUNDARIES



count the number of paths between vertices and boundaries

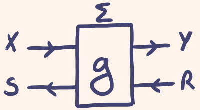
SUMMARY OF THE PROP OF OPEN GRAPHS



OUTLINE

- Games on graphs
- The prop of open graphs
- Games on graphs as functors

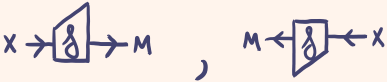
OPEN GAMES



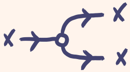
- $P_g : \Sigma \times X \longrightarrow Y$ ⇝ next move
- $C_g : \Sigma \times X \times R \longrightarrow S$ ⇝ coutility
- $B_g : X \times (Y \to R) \longrightarrow \mathcal{P}(\Sigma)$ ⇝ equilibria

[Ghani, Hedges, Winschel, Zahn, Compositional game theory, 2018]

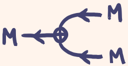
EXAMPLES OF OPEN GAMES



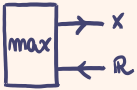
$\rightsquigarrow$ a function



$\rightsquigarrow$ propagating the information



$\rightsquigarrow$ aggregating the information

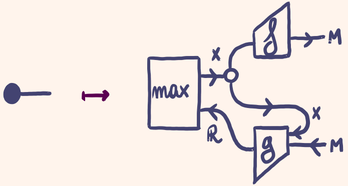


$\rightsquigarrow$ a utility-maximising player

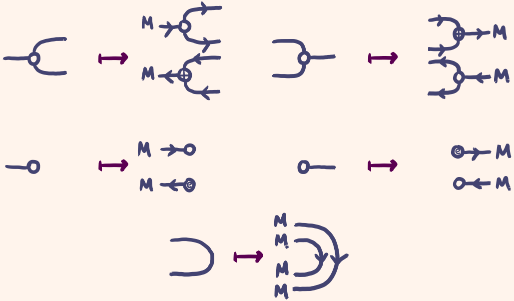
MONOID NETWORK GAMES AS FUNCTORS

$(M, \oplus, e, f: X \to M, g: X \times M \to \mathbb{R})$ monoid network game

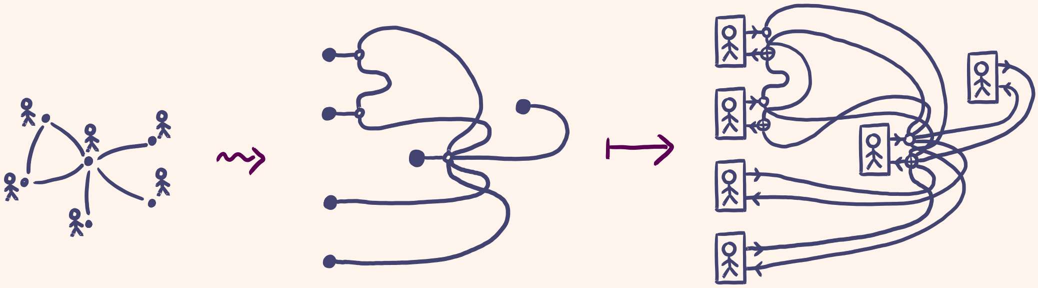
- vertex $\mapsto$ player

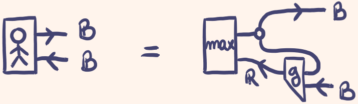


- structure $\mapsto$ structure



THE TRAGEDY OF THE COMMONS





$$g(x,y) = \begin{cases} 1-c+\varepsilon & x,y=0 \\ 1 & y=1,\ x=0 \\ 1-c & x=1 \end{cases}$$

A STANDARD PROCEDURE GRAPHS → GAMES

graph specified with a matrix

$\downarrow$

graph as a morphism in Graph

$\downarrow$ ⟵ monoid network game

game played on the graph as a morphism in Game

SUMMARY

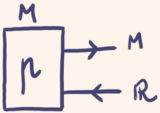
- The category of graphs with boundaries is the free prop generated by

+ equations

- Monoid network games correspond to functors

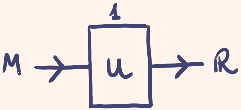
$$\mathcal{G}raph \longrightarrow \mathcal{G}ame$$

EXAMPLE : A PLAYER



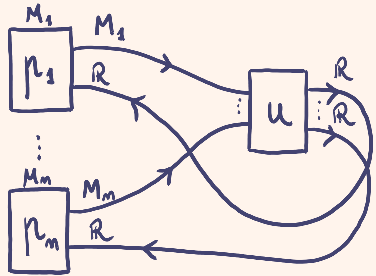
- $P_p(m) = m$
- $C_p(r) = *$
- $B_p(u : M \to \mathbb{R}) = \operatorname{argmax}_{m \in M} u(m)$

EXAMPLE : A (UTILITY) FUNCTION



- $P_u(m) = u(m)$
- $C_u(m) = *$
- $B_u(m) = \{*\}$

NETWORK GAMES IN OPEN GAMES



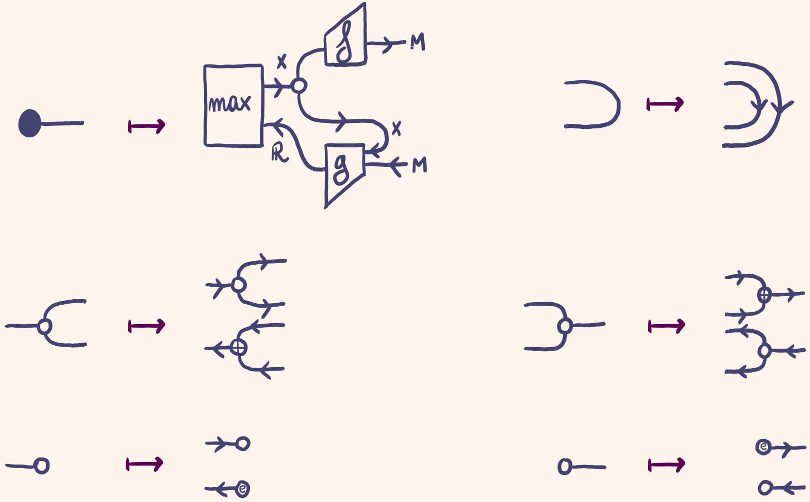
- $p_i : \binom{1}{1} \xrightarrow{M_i} \binom{M_i}{\mathbb{R}}$ ⇝ players
- $u : \binom{M_1 \times \dots \times M_n}{1} \xrightarrow{1} \binom{\mathbb{R}^n}{1}$ ⇝ utility function

MONOID NETWORK GAMES

- $(M, \oplus, e)$ monoid
- X set
- $f : X \longrightarrow M$
- $g : X \times M \longrightarrow \mathbb{R}$

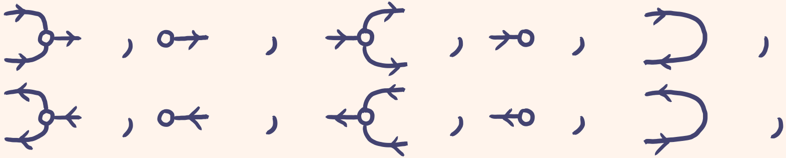
$$u_i(G; x_1, \dots, x_n) = g\Big(x_i, \bigoplus_{j \text{ neighbours of } i} f(x_j)\Big)$$

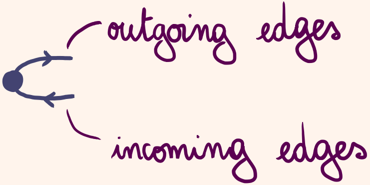
MONOID NETWORK GAMES AS FUNCTORS



DIRECTED OPEN GRAPHS

Two generators for the objects: $\to$, $\leftarrow$





CHANGING THE INCENTIVES