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EFFECTFUL TRANSITION SYSTEMS

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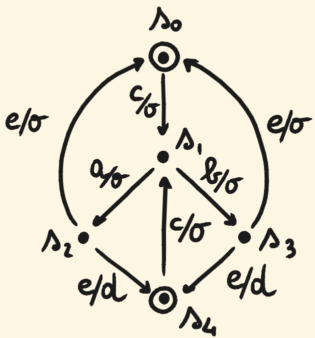
OVERVIEW

- effectful transition systems
- bisimulation
- trace equivalence

CLASSICAL TRANSITION SYSTEMS

A transition system with state space S, inputs A and outputs B is a relation $f \subseteq S \times A \times S \times B$ with a set $S_0 \subseteq S$ of initial states.



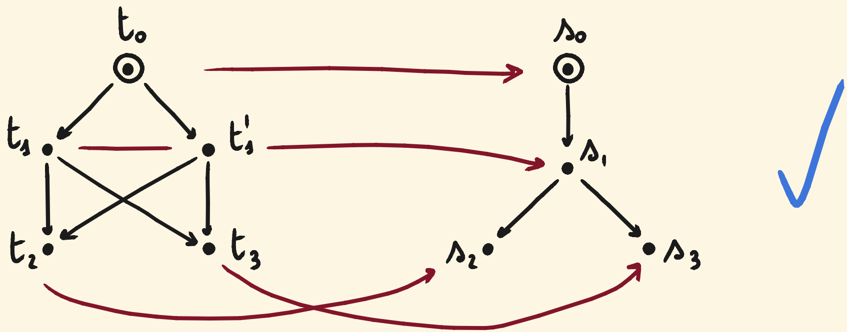
$S = \{s_0, s_1, s_2, s_3, s_4\}$
$A = \{a, b, c, e\}$
$B = \{d, \sigma\}$
$S_0 = \{s_0, s_4\}$

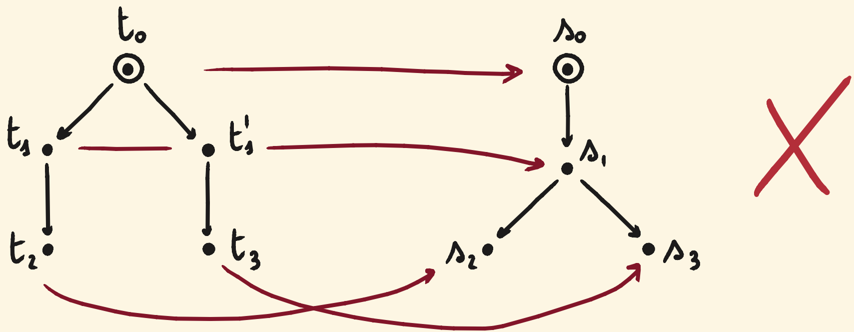
MORPHISMS OF TRANSITION SYSTEMS

A morphism of transition systems $u : (f, S, S_0) \to (g, T, T_0)$ is a function $u : S \to T$ such that

- for all $s, s' \in S$, $a \in A$, $b \in B$
  if $(s, a, s', b) \in f$, then $(u(s), a, u(s'), b) \in g$, and,
- for all $s \in S$, $a \in A$, $b \in B$, $t \in T$
  if $(u(s), a, t, b) \in g$, then there is $s' \in S$ with
  $(s, a, s', b) \in f$ and $u(s') = t$
- for all $t \in T_0$ there is $s \in S_0$ with $u(s) = t$

MORPHISMS - EXAMPLE

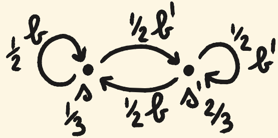




PROBABILISTIC TRANSITION SYSTEMS

A probabilistic transition system with state space S, inputs A and outputs B is a stochastic channel $f : S \times A \to S \times B$ with an initial distribution s_0 on S:

- a function $f : S \times A \times S \times B \to [0,1]$ such that $\{(s',b) \mid f(s,a,s',b) > 0\}$ is finite and $\sum_{s',b} f(s,a,s',b) = 1$
- a function $s_0 : S \to [0,1]$ such that $\{s \mid s_0(s) > 0\}$ is finite and $\sum_{s} s_0(s) = 1$



$S = \{s, s'\}$

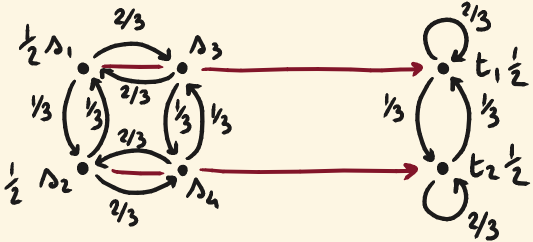
$A = \{*\}$

$B = \{b, b'\}$

MORPHISMS OF PROBABILISTIC TRANSITION SYSTEMS

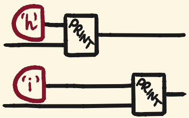
A morphism of transition systems $u : (f, S, s_0) \to (g, T, t_0)$ is a function $u : S \to T$ such that

- $\sum_{\substack{s' \\ u(s')=t'}} f(s', b \mid s, a) = g(t', b \mid u(s), a)$
- $\sum_{\substack{s \\ u(s)=t}} s_0(s) = t_0(t)$

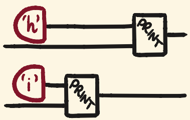


EFFECTFUL CATEGORIES

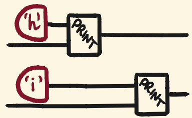
An effectful category $(\mathcal{V},\mathcal{C})$ is an identity on objects monoidal functor $V : \mathcal{V} \to \mathcal{Z}(\mathcal{C})$ from a monoidal category $\mathcal{V}$ to the centre $\mathcal{Z}(\mathcal{C})$ of a premonoidal category $\mathcal{C}$.



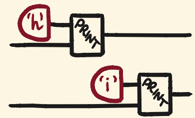
$\neq$



but



$=$



effectful morphisms do not interchange

pure morphisms interchange

[Levy 2004, Román 2022]

EFFECTFUL TRANSITION SYSTEMS

$\mathcal{C}$ effectful category over $\mathcal{S}et$

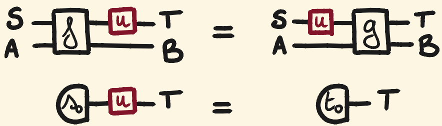
A <u>transition system</u> $(f, S, s_0) : A \to B$ in $\mathcal{C}$ with state space S, inputs A and outputs B is a morphism

$$f : S \otimes A \to S \otimes B$$

with an initial state

$$s_0 : I \to S$$

A <u>morphism</u> of transition systems $u : (f, S, s_0) \to (g, T, t_0)$ is a function $u : S \to T$ such that



[cf. Katis, Sabadini, Walters 1997]

EFFECTFUL TRANSITION SYSTEMS - EXAMPLES

- $\mathcal{C} = \mathrm{Rel}$
  $\Rightarrow$ classical transition systems and their morphisms

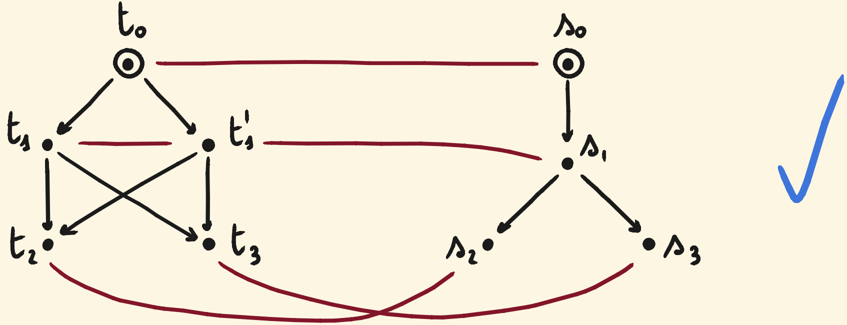
- $\mathcal{C} = \mathrm{Kl}(\mathcal{D})$
  $\Rightarrow$ probabilistic transition systems and their morphisms

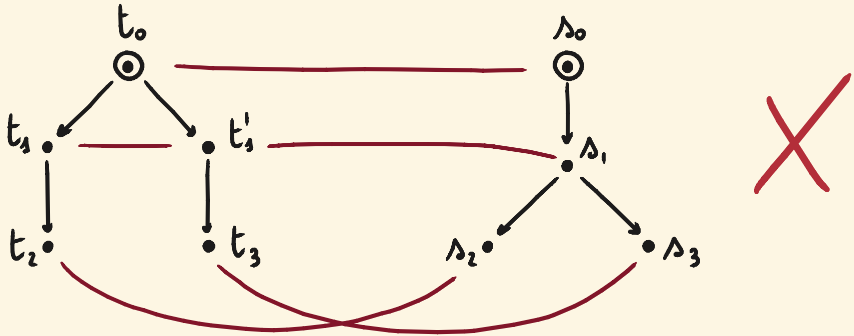
BISIMULATION

Two transition systems (f, S, S_0) and (g, T, T_0) are bisimilar if there is a relation $R \subseteq S \times T$ such that

- for all $s, s' \in S$, $t \in T$, $a \in A$, $b \in B$
  if $(s,t) \in R$ and $(s,a,s',b) \in f$, then there is $t' \in T$ with $(s',t') \in R$ and $(t,a,t',b) \in g$
  if $s' \in S_0$, then there is $t' \in T_0$ with $(s',t') \in R$
- for all $s \in S$, $t, t' \in T$, $a \in A$, $b \in B$
  if $(s,t) \in R$ and $(t,a,t',b) \in g$, then there is $s' \in S$ with $(s',t') \in R$ and $(s,a,s',b) \in f$
  if $t' \in T_0$, then there is $s' \in S_0$ with $(s',t') \in R$

BISIMULATION - EXAMPLE





CHARACTERISING BISIMULATION

PROPOSITION (Rutten, 1995)
Two transition systems (f, S, S_0) and (g, T, T_0) are bisimilar iff there is a span $(f, S, S_0) \xleftarrow{u} (h, R, R_0) \xrightarrow{v} (g, T, T_0)$ of morphisms of transition systems.

⇝ R is a bisimulation and u, v are projections

PROBABILISTIC BISIMULATION

Two transition systems (f, S, s_0) and (g, T, t_0) are bisimilar if there is a relation $R \subseteq S \times T$ such that

- for all $s \in S$, $t \in T$, $a \in A$, $b \in B$, X equivalence class
  if $(s,t) \in R$ $\displaystyle\sum_{s' \in X \cap S} f(s', b \mid s, a) = \sum_{t' \in X \cap T} g(t', b \mid t, a)$

- for all equivalence classes X
  $$\sum_{s \in X \cap S} s_0(s) = \sum_{t \in X \cap T} t_0(t)$$

[Blute, Desharnais, Edalat, Panangaden 1997]

EFFECTFUL BISIMULATION

Two effectful transition systems $(f,S),(g,T) : A \to B$ are <u>bisimilar</u> if they are connected by spans of morphisms:

$$(f,S) \leftarrow (h_1,R_1) \rightarrow (f_1,S_1) \leftarrow (h_2,R_2) \cdots (h_n,R_n) \rightarrow (g,T)$$

PROPOSITION

When $\mathcal{C} = Kl(M)$, then (f,S) and (g,T) are bisimilar iff they have the same bisimulation quotient, i.e. there is (h,Q) with morphisms $(f,S) \xrightarrow{r} (h,Q) \xleftarrow{q} (g,T)$.

EFFECTFUL BISIMULATION - EXAMPLES

- $\mathcal{C} = \mathrm{Rel}$
  $\Rightarrow$ classical bisimulation

- $\mathcal{C} = \mathrm{Kl}(\mathcal{D})$
  $\Rightarrow$ probabilistic bisimulation

EXECUTION TRACES

A <u>trace</u> of a transition system (δ, S, S_0) relative to a sequence $(a_0, a_1, \dots)$ of inputs is a sequence $(b_0, b_1, \dots)$ of outputs that has a sequence $(s_0, s_1, \dots)$ of states generating it:

- $s_0 \in S_0$
- for all n, $(s_n, a_n, s_{n+1}, b_n) \in \delta$

Two transition systems are <u>trace equivalent</u> if, for all sequences of inputs, their traces coincide.

PROBABILISTIC EXECUTION TRACES

A *trace* of a transition system (f, S, s_0) relative to a sequence $(a_0, a_1, \ldots)$ of inputs is a sequence $(d_0, d_1, \ldots)$ of distributions d_n on B^{n+1}

- $f_0(s', b_0) = \sum_{s \in S} f(s', b_0 \mid s, a_0) \cdot s_0(s)$
- $f_{n+1}(s', b_{n+1}, \ldots, b_0) = \sum_{s \in S} f(s', b_{n+1} \mid s, a_{n+1}) \cdot f_n(s, b_n, \ldots, b_0)$

and $d_n(b_{n+1}, \ldots, b_0) = \sum_{s} f_n(s, b_{n+1}, \ldots, b_0)$

Two transition systems are *trace equivalent* if, for all sequences of inputs, their traces coincide.

[Blute, Desharnais, Edalat, Panangaden 1997]

EFFECTFUL STREAMS

An effectful stream $f : \mathbb{A} \to \mathbb{B}$ with inputs $\mathbb{A} = (A_0, A_1, \dots)$ and outputs $\mathbb{B} = (B_0, B_1, \dots)$ is

- a memory $M_f \in \mathrm{obj}\,\mathcal{C}$
- a first action $\mathrm{now}(f) : A_0 \to M_f \otimes B_0$ in $\mathcal{C}$
- a rest of the action $\mathrm{later}(f) : M_f \cdot \mathbb{A}^+ \to \mathbb{B}^+$

quotiented by the equivalence relation generated by

$$f \sim g \text{ if there is a function } u \quad \begin{cases} \mathrm{now}\, f = \mathrm{now}\, g \,;\, (u \otimes 1) \\ u \cdot \mathrm{later}\, f \sim \mathrm{later}\, g \end{cases}$$

[DL, de Felice, Román 2022]

EFFECTFUL TRACES

The <u>trace</u> of an effectful transition system (f, S, s_0) is the effectful stream $tr(f, S, s_0)$ defined coinductively by

- $now(tr(f, S, s_0)) := (s_0 \otimes \mathbb{1}) ; f$
- $later(tr(f, S, s_0)) := (\!| f |\!)$

where $(\!| f |\!)$ is the effectful stream

- $now(\!| f |\!) := f$
- $later(\!| f |\!) := (\!| f |\!)$

Two effectful transition systems (f, S, s_0) and (g, T, t_0) are <u>trace equivalent</u> if $tr(f, S, s_0) = tr(g, T, t_0)$.

EFFECTFUL TRACES - EXAMPLES

- $\mathcal{C} = \mathrm{Rel}$
  $\Rightarrow$ classical trace equivalence

- $\mathcal{C} = \mathrm{Kl}(\mathcal{D})$
  $\Rightarrow$ probabilistic trace equivalence

BISIMULATION ⇒ TRACE EQUIVALENCE

PROPOSITION

If two effectful transition systems are bisimilar, then they are trace equivalent.

PROOF SKETCH

$u : (f, S, s_0) \to (g, T, t_0)$

$$\begin{aligned}
\Rightarrow \text{now}\, tr(g, T, t_0) &= (t_0 \otimes \mathbb{1}) ; g \\
&= ((s_0 ; u) \otimes \mathbb{1}) ; g \\
&= (s_0 \otimes \mathbb{1}) ; f ; (u \otimes \mathbb{1}) \\
&= \text{now}\, tr(f, S, s_0) ; (u \otimes \mathbb{1})
\end{aligned}$$

by coinduction, $u \cdot \text{later}\, tr(g, T, t_0) \sim \text{later}\, tr(f, S, s_0)$ □

CONCLUSIONS & FUTURE WORK

- transition systems, bisimulation and trace equivalence in effectful categories

- premonoidal examples
- internal language of premonoidal categories
- string diagrams in $\mathrm{St}\,\mathcal{C}$ are sound for bisimulation, can we get completeness?