Directions and perspectives in the λ-calculus – 8 January 2024

EFFECTFUL STREAMS FOR EFFECTFUL TRACE EQUIVALENCE

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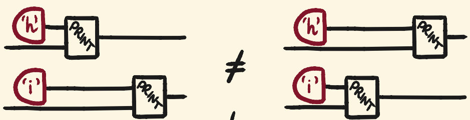
OVERVIEW

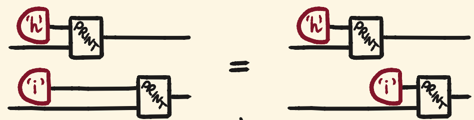
Effectful streams
give categorical semantics to
effectful Mealy machines.

EFFECTFUL CATEGORIES

An effectful category $(\mathcal{V}, \mathcal{C})$ is an identity on objects monoidal functor $V : \mathcal{V} \to \mathcal{Z}(\mathcal{C})$ from a monoidal category $\mathcal{V}$ to the centre $\mathcal{Z}(\mathcal{C})$ of a premonoidal category $\mathcal{C}$.



but



effectful morphisms do not interchange

pure morphisms interchange

[Levy 2004, Román 2022]

OVERVIEW

[• Effectful streams]
• Effectful traces
• Effectful bisimulation

EFFECTFUL STREAMS

An effectful stream $\mathbb{F} : \mathbb{A} \to \mathbb{B}$ with inputs $\mathbb{A} = (A_0, A_1, \dots)$ and outputs $\mathbb{B} = (B_0, B_1, \dots)$ is

- a memory $M_{\mathbb{F}} \in \mathrm{obj}\,\mathcal{C}$
- a first action $\mathrm{now}(\mathbb{F}) : A_0 \to M_{\mathbb{F}} \otimes B_0$ in $\mathcal{C}$
- a rest of the action $\mathrm{later}(\mathbb{F}) : M_{\mathbb{F}} \cdot \mathbb{A}^+ \to \mathbb{B}^+$

quotiented by the equivalence relation generated by

$\mathbb{F} \sim \mathbb{G}$ if there is a pure morphism $u : M_{\mathbb{F}} \to M_{\mathbb{G}}$ st

$\mathrm{now}\,\mathbb{F} = \mathrm{now}\,\mathbb{G} ; (u \otimes 1)$ and $u \cdot \mathrm{later}\,\mathbb{F} \sim \mathrm{later}\,\mathbb{G}$

[cf. DL, de Felice, Román (2022) Monoidal streams for dataflow programming]

CARTESIAN STREAMS

$\mathcal{V} = \mathcal{C}$ cartesian

PROPOSITION

A cartesian stream $F : \mathbb{A} \to \mathbb{B}$ with inputs $\mathbb{A} = (A_0, A_1, \dots)$ and outputs $\mathbb{B} = (B_0, B_1, \dots)$ is a family of morphisms $f_n : A_0 \times \dots \times A_n \longrightarrow B_n$, for $n \in \mathbb{N}$.

THEOREM

$\mathrm{Stream}\,\mathcal{C} \simeq \mathrm{coKl}\ \mathrm{List}^+$ iff $\mathcal{C}$ cartesian.

[Uustalu, Vene (2005) The essence of dataflow programming]

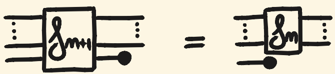
STOCHASTIC STREAMS

$\mathcal{V} = \mathrm{Kl}\,\mathcal{D}$

$\mathcal{C} = \mathrm{Kl}\,\mathcal{D}$

THEOREM

A stochastic stream $\mathbb{F} : \mathbb{A} \to \mathbb{B}$ with inputs $\mathbb{A} = (A_0, A_1, \dots)$ and outputs $\mathbb{B} = (B_0, B_1, \dots)$ is a family of functions $f_n : A_0 \times \dots \times A_n \longrightarrow \mathcal{D}(B_0 \times \dots \times B_n)$, for $n \in \mathbb{N}$, such that



in $\mathrm{Kl}\,\mathcal{D}$.

[DL, de Felice, Román (2022) Monoidal streams for dataflow programming]

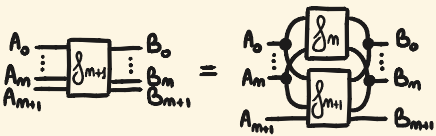
RELATIONAL STREAMS

$\mathcal{V} = \mathbf{Rel}_{TOT}$

$\mathcal{C} = \mathbf{Rel}$

THEOREM

A relational stream $\mathbb{F} : \mathbb{A} \to \mathbb{B}$ with inputs $\mathbb{A} = (A_0, A_1, \dots)$ and outputs $\mathbb{B} = (B_0, B_1, \dots)$ is a family of functions $f_n : A_0 \times \dots \times A_n \longrightarrow \mathcal{P}(B_0 \times \dots \times B_n)$, for $n \in \mathbb{N}$, such that



in Rel

OVERVIEW

- Effectful streams
[• Effectful traces]
- Effectful bisimulation

EFFECTFUL MEALY MACHINES

$\mathcal{C}$ effectful category over $\mathcal{V}$

A Mealy machine $(f, S, s_0) : A \to B$ in $\mathcal{C}$ with state space S, inputs A and outputs B is a morphism

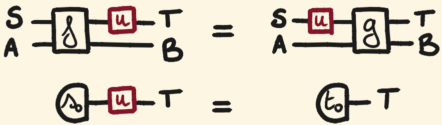
$$f : S \otimes A \to S \otimes B$$

with an initial state

$$s_0 : I \to S$$

A morphism of Mealy machines $u : (f, S, s_0) \to (g, T, t_0)$ is a pure, total and deterministic morphism $u : S \to T$

such that



[cf. Katis, Sabadini, Walters 1997]

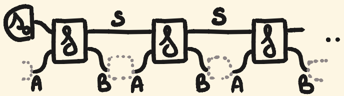
TRACE EQUIVALENCE

The <u>trace</u> of an effectful Mealy machine (f, S, s_0) is the effectful stream $\mathrm{tr}(f, S, s_0)$ defined coinductively by

- $\mathrm{now}(\mathrm{tr}(f, S, s_0)) := (s_0 \otimes \mathbb{1}) ; f$
- $\mathrm{later}(\mathrm{tr}(f, S, s_0)) := (\!| f |\!)$

where $\mathrm{now}(\!| f |\!) := f$ and $\mathrm{later}(\!| f |\!) := (\!| f |\!)$.

$\rightsquigarrow$ $\mathrm{tr}(f, S, s_0) =$



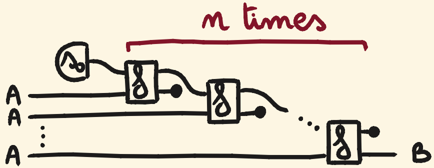
Two effectful Mealy machines (f, S, s_0) and (g, T, t_0) are <u>trace equivalent</u> if $\mathrm{tr}(f, S, s_0) = \mathrm{tr}(g, T, t_0)$.

CARTESIAN TRACES

$\mathcal{V} = \mathcal{C}$ cartesian

$(f, S, s_0) : A \to B$

$\mathrm{tr}(f, S, s_0)_n : A^n \to B$

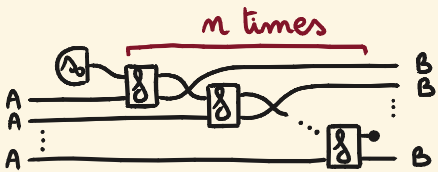


STOCHASTIC TRACES

$\mathcal{V} = \mathcal{S}et$
$\mathcal{C} = \mathcal{K}\ell\,\mathcal{D}$

$(f, S, s_0) : A \to B$

$tr(f, S, s_0)_n : A^n \longrightarrow \mathcal{D}(B^n)$



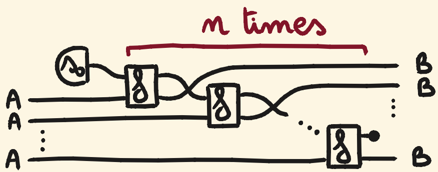
in $\mathcal{K}\ell\,\mathcal{D}$

RELATIONAL TRACES

$\mathcal{V} = \mathcal{S}et$
$\mathcal{C} = \mathcal{R}el$

$$(f, S, s_0) : A \to B$$

$$tr(f, S, s_0)_n : A^n \longrightarrow P(B^n)$$



in $\mathcal{R}el$

OVERVIEW

- Effectful streams
- Effectful traces
- [• Effectful bisimulation]

BISIMULATION

Two effectful Mealy machines $(f,S,s),(g,T,t) : A \to B$ are <u>bisimilar</u> if they belong to the same connected component in $\mathrm{MMach}(A,B)$:

$$(f,S,s) \xleftarrow{u_1} (h_1,R_1,r_1) \xrightarrow{v_1} (f_1,S_1,s_1) \xleftarrow{u_2} \cdots (h_n,R_n,r_n) \xrightarrow{v_n} (g,T,t)$$

PROPOSITION

When $\mathscr{C} = Kl(M)$, then (f,S,s) and (g,T,t) are bisimilar iff they have the same bisimulation quotient, i.e. there is (h,Q,q) with morphisms

$$(f,S,s) \xrightarrow{u} (h,Q,q) \xleftarrow{v} (g,T,t) .$$

PROBABILISTIC BISIMULATION

$$\begin{cases} f : S \times A \to \mathcal{D}(S \times B) \\ s_0 : I \to \mathcal{D}(S) \end{cases} \qquad \begin{cases} g : T \times A \to \mathcal{D}(T \times B) \\ t_0 : I \to \mathcal{D}(T) \end{cases}$$

$R \subseteq S \times T$ is a bisimulation if there are $\begin{cases} u : S \to Q \\ v : T \to Q \end{cases}$ st

- $sRt \Leftrightarrow u(s) = v(t)$
- $sRt \Rightarrow \forall q \in Q,\ a \in A,\ b \in B$

$$\sum_{s' \in u^{-1}(q)} f(s', b \mid s, a) = \sum_{t' \in v^{-1}(q)} g(t', b \mid t, a)$$

⇝ there is a quotient Mealy machine

[Larsen, Skou (1991), Blute, Desharnais, Edalat, Panangaden (1997)]

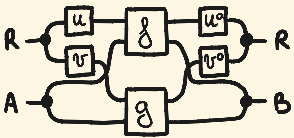
RELATIONAL BISIMULATION

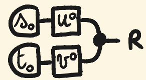
$\begin{cases} f : S \times A \to P(S \times B) \\ s_0 : I \to P(S) \end{cases}$ $\begin{cases} g : T \times A \to P(T \times B) \\ t_0 : I \to P(T) \end{cases}$

$R \subseteq S \times T$ is a bisimulation if

PROPOSITION

$R \subseteq S \times T$ is a bisimulation between (f, S, s_0) and (g, T, t_0) iff the projections $u : R \to S$ and $v : R \to T$ are morphisms $u : (f, S, s_0) \to (h, R, r_0)$ and $v : (g, T, t_0) \to (h, R, r_0)$

for $h :=$ 

and $r_0 :=$ 

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[Rutten (1995)]

BISIMULATION $\Rightarrow$ TRACE EQUIVALENCE

PROPOSITION

If two effectful Mealy machines are bisimilar, then they are trace equivalent.

PROOF SKETCH

$u : (f, S, s_0) \to (g, T, t_0)$

$$\begin{aligned}
\Rightarrow \text{now}\ tr(g, T, t_0) &= (t_0 \otimes \mathbb{1}) ; g \\
&= ((s_0 ; u) \otimes \mathbb{1}) ; g \\
&= (s_0 \otimes \mathbb{1}) ; f ; (u \otimes \mathbb{1}) \\
&= \text{now}\ tr(f, S, s_0) ; (u \otimes \mathbb{1})
\end{aligned}$$

by coinduction, $u \cdot \text{later}\ tr(g, T, t_0) \sim \text{later}\ tr(f, S, s_0)$ $\square$

CONCLUSIONS & FUTURE WORK

- Effectful streams
- Mealy machines, bisimulation and trace equivalence in effectful categories

- premonoidal examples
- internal language of premonoidal categories
- string diagrams in $St\,\mathcal{C}$ are sound for bisimulation, can we get completeness?