Topos colloquium 16 May 2024

EFFECTFUL TRACE SEMANTICS VIA EFFECTFUL STREAMS

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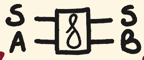
MEALY MACHINES

Systems are $\delta : S \otimes A \to S \otimes B$ with $s_0 : I \to S$

state space



input / output alphabets

initial state

- native sequential and parallel compositions
- parametric in the underlying process theory
- premonoidal categories for global effects

⇝ what is their behaviour ?
when are two of them equivalent ?

[cf. Katis, Sabadini, Walters 1997]

COALGEBRAIC SEMANTICS

Systems are coalgebras $f : S \to F(S)$

input/output

$S \to (S \times B)^A$

non-determinism

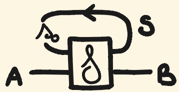
$S \to \mathcal{P}(S \times B)$

- bisimulation is equality in the final coalgebra

⇝ how do these compose?
how to change the underlying process theory?

OVERVIEW

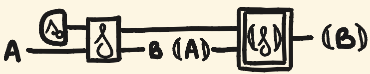
effectful Mealy machines



free construction

~ syntax

effectful streams



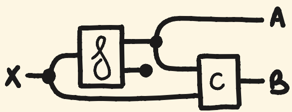
coalgebraic construction

~ semantics

STRING DIAGRAMS & DO-NOTATION

- Symmetric monoidal categories are theories of processes
- String diagrams and do-notation are convenient syntax

$$\nu_X ; ((f ; (\nu_A \otimes \varepsilon_B)) \otimes 1\!\!1_X) ; (1\!\!1_A \otimes c)$$

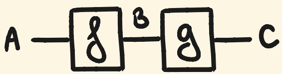


cond(x) = do
  f(x) → (a, b)
  c(a, x) → b'
  return (a, b')

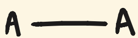
STRING DIAGRAMS & DO-NOTATION

- resources $A, B, \dots$ and processes $f : A \to B$, with possibly multiple inputs/outputs $h : A \to B \otimes B'$, $s : I \to B$

- sequential composition $f ; g : A \to C$ and identities $\mathbb{1}_A : A \to A$



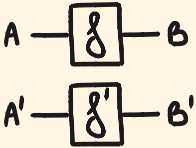

```
comp f g (a) = do
  f(a) → b
  g(b) → c
  return (c)
```




```
id (a) = do
  return (a)
```

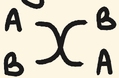
STRING DIAGRAMS & DO-NOTATION

- parallel composition $f \otimes f' : A \otimes A' \to B \otimes B'$

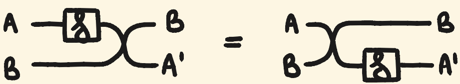



```
tensor f f' (a, a') = do
  f(a) → b
  f'(a') → b'
  return (b, b')
```

- permuting resources $\sigma_{A,B} : A \otimes B \to B \otimes A$

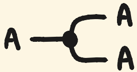



```
swap (a, b) = do
  return (b, a)
```




```
swapNat (a, b) = do
  f(a) → a'
  return (b, a')
```

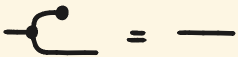
COPY AND DISCARD



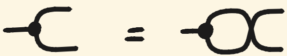

```
copy(a) = do
  return(a,a)
```


```
discard(a) = do
  return()
```

```
copyAssoc(a) = do
  return(a,a,a)
```



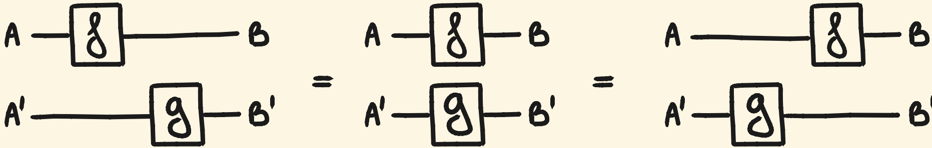
```
copyUnit(a) = do
  return(a)
```



```
copyCommut(a) = do
  return(a,a)
```

[Corradini & Gadducci 1999]

THE INTERCHANGE LAW




```
par f g (a, a') = do
  f(a) → b
  g(a') → b'
  return (b, b')
```

=

```
par f g (a, a') = do
  g(a') → b'
  f(a) → b
  return (b, b')
```

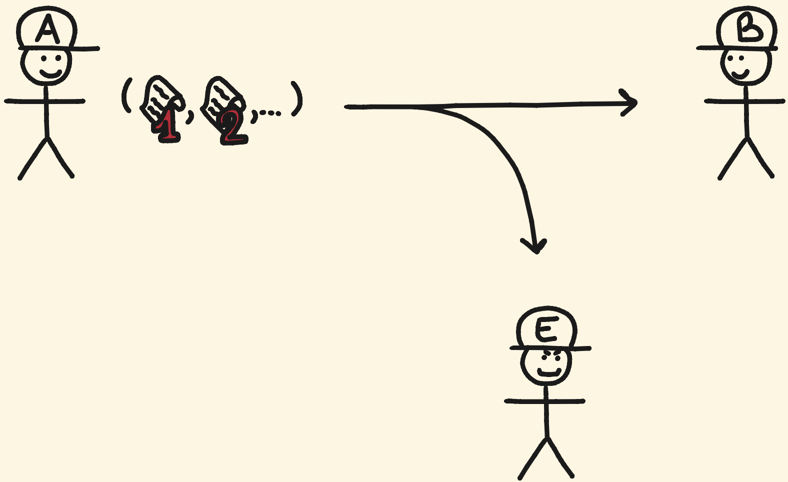
⇝ holds in monoidal categories

OUTLINE

[• effectful copy-discard categories]
- effectful Mealy machines
- effectful streams
- trace semantics
- causal processes
- bisimulation

A MOTIVATING EXAMPLE

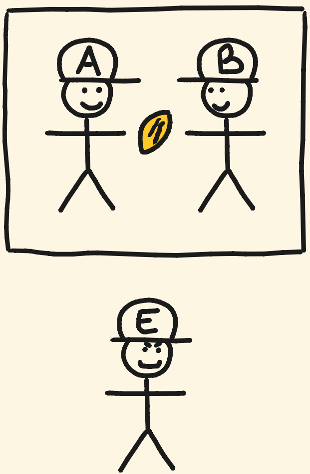
The stream cipher is a simple cryptographic protocol.



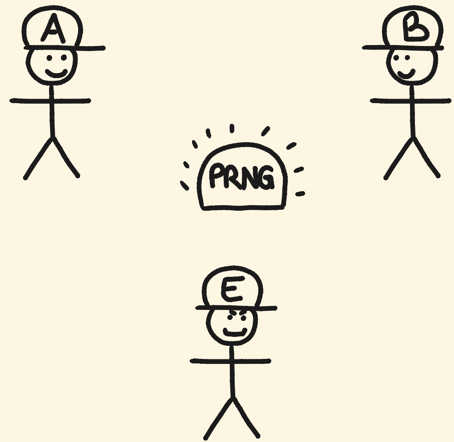
[cf. Broadbent, Karvonen 2022]

STREAM CIPHER PROTOCOL (1)

1. share a seed through a secure channel

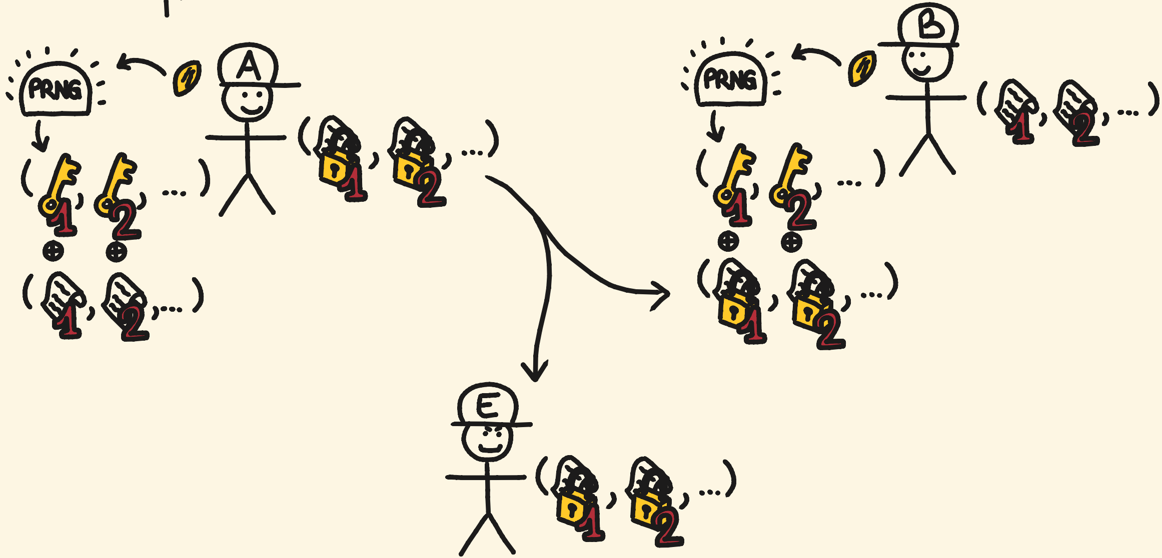


2. share a pseudorandom number generator



STREAM CIPHER PROTOCOL (2)

3. send a stream of encrypted messages through a public channel



COMPUTATIONS WITH EFFECTS

- Stochastic effects: generating the seed

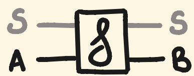
$\mathcal{D} : \mathbf{Set} \to \mathbf{Set}$ distribution monad

$\mathcal{D}(A) := \{\sigma : A \to [0,1] \mid \operatorname{supp}\sigma \text{ is finite} \wedge \sum_{a \in A} \sigma(a) = 1\}$

- Global state: sharing the seed

$\mathrm{State}_S : \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Set}$ state promonad

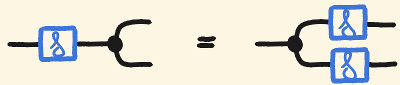
$\mathrm{State}_S(A,B) := \mathcal{C}(S \otimes A, S \otimes B)$



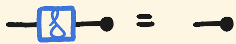
VALUES

Values are both :

- deterministic

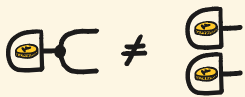


- total

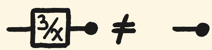


ex $(3\cdot -) : \mathbb{R} \longrightarrow \mathbb{R}$

NON-ex Flip : $1 \longrightarrow \mathcal{D}(\{H,T\})$

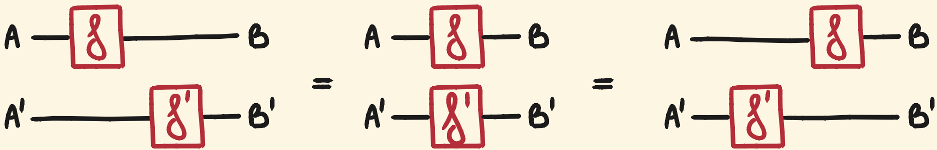


$(3/-) : \mathbb{R} \longrightarrow \mathbb{R}$



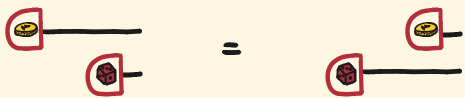
LOCAL COMPUTATIONS

local computations interchange,



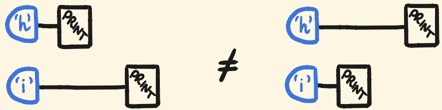
$$\mathrm{localF}(a, a') = \mathrm{do} \quad \begin{array}{|l} f(a) \to b \\ f'(a') \to b' \\ \mathrm{return}\,(b, b') \end{array} \quad = \quad \mathrm{localF}(a, a') = \mathrm{do} \quad \begin{array}{|l} f'(a') \to b' \\ f(a) \to b \\ \mathrm{return}\,(b, b') \end{array}$$

<u>ex</u> Stoch



EFFECTFUL COMPUTATIONS

Effectful computations may have global effects.

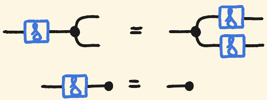


$$
\begin{array}{l}
\mathrm{printHI}(\,) = \mathrm{do} \\
\quad \text{'h'}(\,) \to c_1 \\
\quad \text{'i'}(\,) \to c_2 \\
\quad \mathrm{print}(c_1) \rightsquigarrow (\,) \\
\quad \mathrm{print}(c_2) \rightsquigarrow (\,) \\
\quad \mathrm{return}\ (\,)
\end{array}
\quad \neq \quad
\begin{array}{l}
\mathrm{printIH}(\,) = \mathrm{do} \\
\quad \text{'h'}(\,) \to c_1 \\
\quad \text{'i'}(\,) \to c_2 \\
\quad \mathrm{print}(c_2) \rightsquigarrow (\,) \\
\quad \mathrm{print}(c_1) \rightsquigarrow (\,) \\
\quad \mathrm{return}\ (\,)
\end{array}
$$

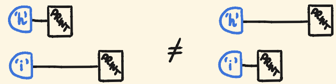
ex state promonads, IO monad

EFFECTFUL COPY-DISCARD CATEGORIES

Values can be copied and discarded (cartesian)

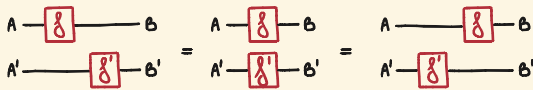


Effectful computations may have global effects (premonoidal)



$$\mathcal{V} \longrightarrow \mathcal{L} \longrightarrow \mathcal{C}$$

Local computations interchange (monoidal)



ex ($\mathcal{S}et$, $\mathcal{S}toch$, $\mathcal{S}tate_S$)

($\mathrm{cart}(\mathcal{C})$, $\mathcal{Z}(\mathcal{C})$, $\mathcal{C}$) for a CD-premonoidal $\mathcal{C}$

OUTLINE

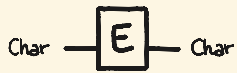
- effectful copy-discard categories
- effectful Mealy machines
- effectful streams
- trace semantics
- causal processes
- bisimulation

STREAM CIPHER COMPONENTS

Encryption protocol

Decryption protocol

Attacker protocol



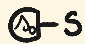
EFFECTFUL MEALY MACHINES

A <u>Mealy machine</u> $(f, S, s_0) : A \to B$ in $(\mathcal{V}, \mathcal{L}, \mathcal{C})$
is a morphism

$$f : S \otimes A \to S \otimes B$$

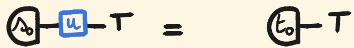
with an initial state

$$s_0 : I \to S$$



A <u>morphism</u> of Mealy machines $u : (f, S, s_0) \to (g, T, t_0)$
is a value morphism $u : S \to T$ in $\mathcal{V}$

such that

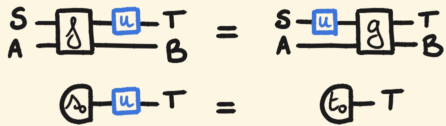


[cf. Katis, Sabadini, Walters 1997 ; EDL, Gianola, Román, Sabadini, Sobociński 2022]

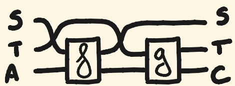
EFFECTFUL CATEGORY OF MEALY MACHINES

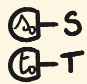
Mealy is an effectful category where

- objects are the objects of $\mathcal{C}$
- morphisms $(f, S, s_0) : A \to B$ are Mealy machines quotiented by value isomorphisms $u : S \xrightarrow{\cong} T$



- composition tensors the state spaces



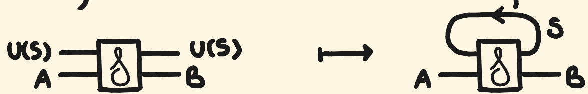


FEEDBACK EFFECTFUL CATEGORIES

A feedback effectful category $\mathcal{C}$ is a premonoidal category $\mathcal{C}$ with

- a monoidal category $\mathcal{S}$
- a premonoidal functor $U : \mathcal{S} \to \mathcal{C}$
- an operation

$$\mathrm{Fbk} : \mathcal{C}(U(S) \otimes A,\ U(S) \otimes B) \longrightarrow \mathcal{C}(A, B)$$

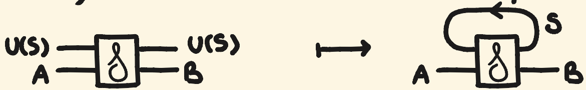


+ axioms

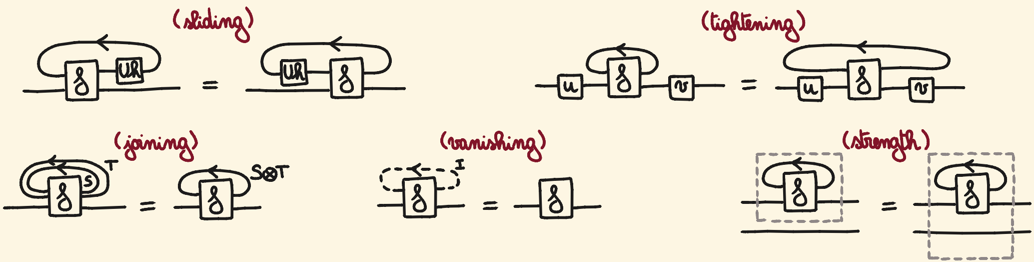
FEEDBACK EFFECTFUL CATEGORIES

U-FEEDBACK

$$\mathrm{Fbk} : \mathcal{C}(U(S) \otimes A, U(S) \otimes B) \longrightarrow \mathcal{C}(A, B)$$



satisfying



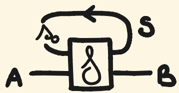
EFFECTFUL CATEGORY OF MEALY MACHINES

$\mathcal{S} := \mathrm{pt}\mathcal{C}_{\mathrm{iso}}$

THEOREM

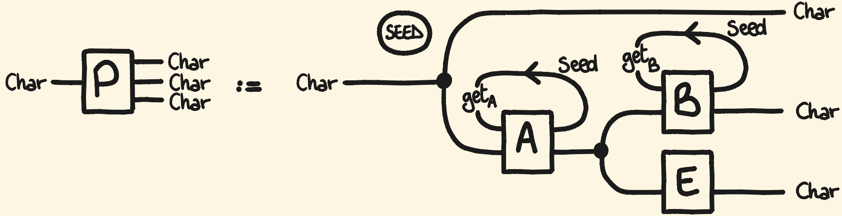
Mealy is the free pointed-feedback category over $\mathcal{C}$.

$$\mathrm{Mealy}(A,B) = \int^{(s_0,S) \in \mathrm{pt}\mathcal{C}_{\mathrm{iso}}} \mathcal{C}(S \otimes A, S \otimes B)$$



[cf. Katis, Sabadini, Walters 1997 ; EDL, Gianola, Román, Sabadini, Sobociński 2022]

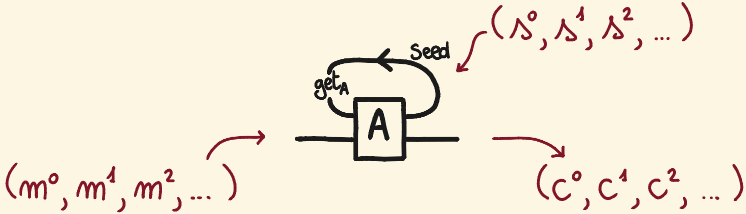
ASSEMBLING THE STREAM CIPHER



OUTLINE

- effectful copy-discard categories
- effectful Mealy machines
- [• effectful streams]
- trace semantics
- causal processes
- bisimulation

EXECUTING MEALY MACHINES



⇝ what should the semantic universe be?
when do two Mealy machines have the same executions?

STREAMS ARE COINDUCTIVE

A stream of elements of A is
- an element $a^0 \in A$
- a stream a^+ of elements of A

⇝ the set of streams is the final coalgebra of the functor

$$A \times (-) : \mathbf{Set} \to \mathbf{Set}$$

PRACTICAL COINDUCTION

Final coalgebras allow

- coinductive definitions

$$\begin{cases} h(x)^\circ := f^\circ(x) \\ h(x)^+ := h(f^+(x)) \end{cases}$$

$$\begin{tikzcd} X \arrow[r, dashed, "\exists! h"] \arrow[d, "(f^\circ, f^+)"'] & \mathrm{stream}_A \arrow[d, "(\_^\circ, \_^+)"] \\ A \times X \arrow[r, dashed, "A \times h"'] & A \times \mathrm{stream}_A \end{tikzcd}$$

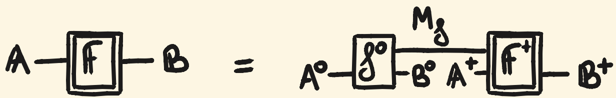
- coinductive proofs

[Kozen & Silva (2017) Practical coinduction]

EFFECTFUL STREAMS

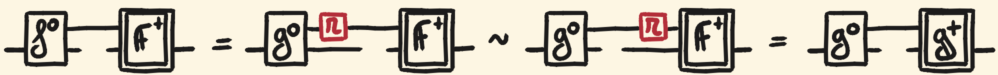
An <u>effectful stream</u> $F : A \to B$ on $(\mathcal{V}, \mathcal{L}, \mathcal{C})$ is

- a memory $M_f \in \mathcal{L}$
- a first action $f^\circ : A^\circ \to M_f \otimes B^\circ$ in $\mathcal{C}$
- the rest of the action $F^+ : M_f \cdot A^+ \to B^+$



quotiented by sliding

$$\begin{cases} f^\circ ; (r \otimes 1) = g^\circ \\ F^+ = r \cdot g^+ \end{cases} \qquad \text{for } r : M_f \to M_g \text{ in } \mathcal{L}$$



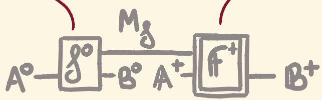
EFFECTFUL STREAMS

The profunctor Stream : $\mathcal{C}^{\mathbb{N}^{op}} \times \mathcal{C}^{\mathbb{N}} \longrightarrow \mathbf{Set}$ is the final coalgebra of the functor

$$F : [\mathcal{C}^{\mathbb{N}^{op}} \times \mathcal{C}^{\mathbb{N}}, \mathbf{Set}] \longrightarrow [\mathcal{C}^{\mathbb{N}^{op}} \times \mathcal{C}^{\mathbb{N}}, \mathbf{Set}]$$

$$F(Q)(A,B) := \int^{M \in \mathcal{C}} \mathcal{C}(A^{\circ}, M \otimes B^{\circ}) \times Q(M \cdot A^{+}, B^{+})$$

quotient by sliding on the memory



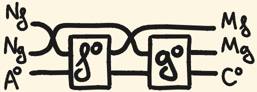
COMPOSITIONAL STRUCTURE OF STREAMS

THEOREM

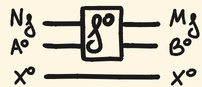
Effectful streams form an effectful category **Stream**.

- composition and monoidal actions are defined coinductively: for $\mathbb{f} : N_f \cdot \mathbb{A} \to \mathbb{B}$ and $\mathbb{g} : N_g \cdot \mathbb{B} \to \mathbb{C}$,

$$\begin{cases} (\mathbb{f} ;_N \mathbb{g})^\circ := \\ (\mathbb{f} ;_N \mathbb{g})^+ := \mathbb{f}^+ ;_M \mathbb{g}^+ \end{cases}$$



$$\begin{cases} (\mathbb{X} \otimes_N \mathbb{f})^\circ := \\ (\mathbb{X} \otimes_N \mathbb{f})^+ := \mathbb{X}^+ \otimes_M \mathbb{f}^+ \end{cases}$$



FEEDBACK ON EFFECTFUL STREAMS

$\partial : \mathsf{Stream} \to \mathsf{Stream}$ $\qquad\qquad \partial(A) := (I, A^{0}, A^{1}, \dots)$

THEOREM

Stream has ∂-feedback.

- feedback is defined coinductively

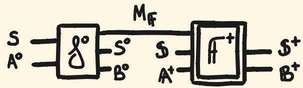
$$\mathbb{F} : (S \cdot \partial \mathbb{S}) \otimes A \longrightarrow \mathbb{S} \otimes \mathbb{B}$$

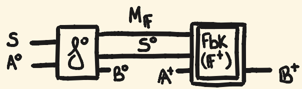
$$\mathsf{Fbk}_{\mathbb{S}} \mathbb{F} : S \cdot A \longrightarrow \mathbb{B}$$

$$M(\mathsf{Fbk}^{S}_{\mathbb{S}} \mathbb{F}) := M(\mathbb{F}) \otimes \mathbb{S}^{\circ}$$

$$(\mathsf{Fbk}^{S}_{\mathbb{S}} \mathbb{F})^{\circ} := f^{\circ}$$

$$(\mathsf{Fbk}^{S}_{\mathbb{S}} \mathbb{F})^{+} := \mathsf{Fbk}^{S^{\circ}}_{\mathbb{S}^{+}} (\mathbb{F}^{+})$$





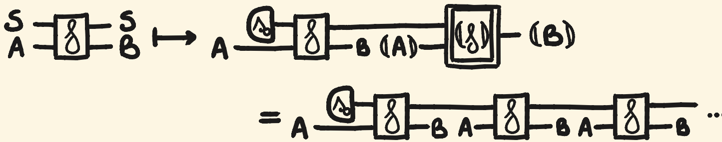
COMPOSITIONAL TRACE SEMANTICS

THEOREM

There is a feedback effectful functor

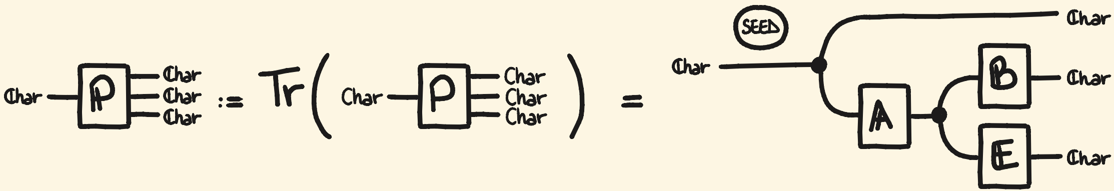
$$Tr : \mathsf{Mealy} \to \mathsf{Stream}$$

$$A \mapsto (A) = (A, A, \dots)$$

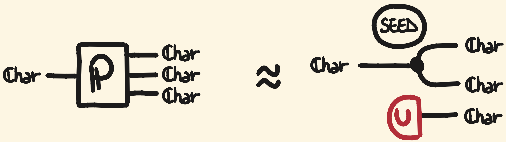


⇝ in Rel these traces coincide with the classical traces

SEMANTICS FOR THE STREAM CIPHER PROTOCOL



SECURITY OF THE PROTOCOL



Alice and Bob see the same message

Eve sees noise

OUTLINE

- effectful copy-discard categories
- effectful Mealy machines
- effectful streams
- trace semantics
- [• causal processes]
- bisimulation

STREAM COMPUTATIONS

- Sliding equivalence might be difficult to handle
- Causal stream functions are old :

  [Raney 1958] shows that they are
  the executions of deterministic Mealy machines

⇝ is there a similar explicit form for effectful streams ?

STREAM COMPUTATIONS

CAUSAL STREAM FUNCTIONS

Stream computations $(p_n)_{n \in \mathbb{N}} : \mathbb{A} \to \mathbb{B}$ in a cartesian category are families $p_n : A_0 \times \cdots \times A_n \longrightarrow B_n$.

STOCHASTIC PROCESSES

Stochastic stream computations $(p_n)_{n \in \mathbb{N}} : \mathbb{A} \to \mathbb{B}$ are families $p_n : A_0 \times \cdots \times A_n \longrightarrow \mathcal{D}(B_0 \times \cdots \times B_n)$ such that $p_n(a_0, \ldots, a_n) = \sum_{a \in A_{n+1}} p_{n+1}(a_0, \ldots, a_n, a)$.

$\rightsquigarrow$ is there a monoidal version ?

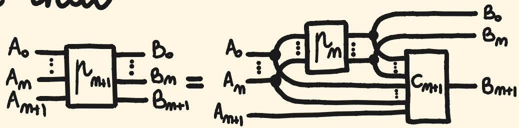
[Sprunger & Katsumata (2019), Uustalu & Vene (2008)]

CAUSAL PROCESSES

A <u>causal process</u> $p : A \to B$ in a copy-discard category $\mathcal{C}$ is a family of morphisms

$$p_n : A_0 \otimes \cdots \otimes A_n \to B_0 \otimes \cdots \otimes B_n$$

such that



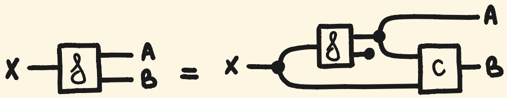
for some $c_{n+1} : B_0 \otimes \cdots \otimes B_n \otimes A_0 \otimes \cdots \otimes A_n \otimes A_{n+1} \to B_{n+1}$

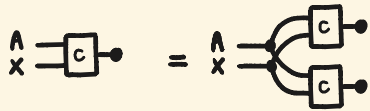
COMPOSING CAUSAL PROCESSES

$\mathcal{C}$ copy-discard

QUASI-TOTAL CONDITIONALS [Fritz (2020), EDL & Román (2023)]

For all $f : X \to A \otimes B$ there is $c : A \otimes X \to B$ st





THEOREM

Causal processes form a monoidal category Proc when $\mathcal{C}$ has quasi-total conditionals.

CAUSAL PROCESSES : EXAMPLES

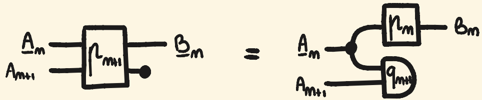
Set

$$p_n : A_0 \times \cdots \times A_n \longrightarrow B_n$$

Par

$$p_n : A_0 \times \cdots \times A_n \longrightarrow (B_0 \times \cdots \times B_n) + 1$$

st



Stoch

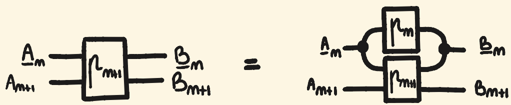
$$p_n : A_0 \times \cdots \times A_n \longrightarrow \mathcal{D}(B_0 \times \cdots \times B_n)$$

st

Rel

$$p_n : A_0 \times \cdots \times A_n \longrightarrow \mathcal{P}(B_0 \times \cdots \times B_n)$$

st

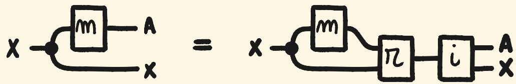


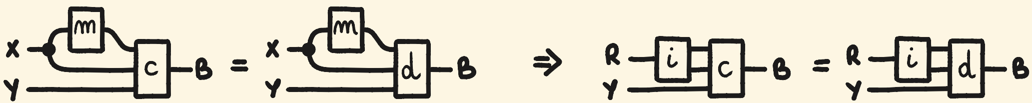
CAUSAL PROCESSES ARE STREAMS

$\mathcal{C}$ copy-discard

RANGES

For all $m : X \to A$ there are $\begin{cases} r : A \otimes X \to R & \text{deterministic} \\ i : R \to A \otimes X & \text{total} \end{cases}$





THEOREM

Consider $(\mathrm{fun}\,\mathcal{C}, \mathrm{tot}\,\mathcal{C}, \mathcal{C})$.

If $\mathcal{C}$ has quasi-total conditionals and ranges,

$$\mathrm{Proc} \simeq \mathrm{Stream}.$$

TRACES ARE EFFECTFUL TRACES

Compute the traces of a Mealy machine

$$(f, S, s) : A \to B$$

in some known cases.

$(b_0, \dots, b_n)$ is a trace of $(a_0, \dots, a_n)$

Set if $s_0 = s$ and $\forall k \leq n \ (s_{k+1}, b_k) = f(s_k, a_k)$

Rel if $\exists (s_0, \dots, s_{n+1}) \ s_0 \in s$ and $\forall k \leq n \ (s_{k+1}, b_k) \in f(s_k, a_k)$

pStoch with probability $\sum_{(s_0, \dots, s_{n+1})} s(s_0 | *) \cdot \prod_{k \leq n} f(s_{k+1}, b_k | s_k, a_k)$

OUTLINE

- effectful copy-discard categories
- effectful Mealy machines
- effectful streams
- trace semantics
- causal processes

[• bisimulation]

COALGEBRAIC BISIMULATION

A bisimulation is a span of coalgebras.

$$\begin{tikzcd} S \arrow[d, "f"'] & R \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\alpha"] & T \arrow[d, "g"] \\ F(S) & F(R) \arrow[l, "F(\pi_1)"] \arrow[r, "F(\pi_2)"'] & F(T) \end{tikzcd}$$

THEOREM [Rutten (2000)]

When $F : \mathbf{Set} \to \mathbf{Set}$ preserves weak pullbacks, bisimularity is an equivalence relation.

[Aczel & Mendler (1989), Rutten (2000)]

BISIMULATION

Two effectful Mealy machines $(f, S, s), (g, T, t) : A \to B$ are <u>bisimilar</u> if they belong to the same connected component in $\mathrm{Mealy}(A, B)$:

$$\begin{tikzcd}
& (h_1, R_1, r_1) \arrow[dl, "u_1"'] \arrow[dr, "v_1"] & & \cdots \arrow[dl, "u_2"'] & (h_n, R_n, r_n) \arrow[dr, "v_n"] & \\
(f, S, s) & & (f_1, S_1, s_1) & & & (g, T, t)
\end{tikzcd}$$

THEOREM

For Mealy machines in $(\mathcal{V}, \mathcal{L}, \mathcal{C})$, bisimulation implies trace equivalence.

PROOF: By coinduction. $\square$

COALGEBRAIC BISIMULATION

PROPOSITION

When $\mathcal{C} = \mathrm{Kl}(M)$, for a commutative monad preserving weak pullbacks, effectful bisimulation coincides with coalgebraic bisimulation.

$(f, S, s) \approx (g, T, t)$ iff
$$\begin{tikzcd}
 & (h, R, r) \arrow[dl, "u"'] \arrow[dr, "v"] & \\
(f, S, s) & & (g, T, t)
\end{tikzcd}$$

EXAMPLES

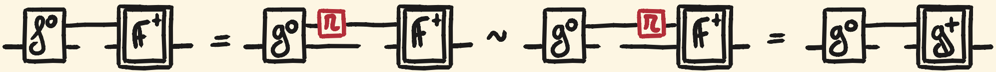
- Set
- Rel
- Stoch
- Par
- pStoch

SUMMARY

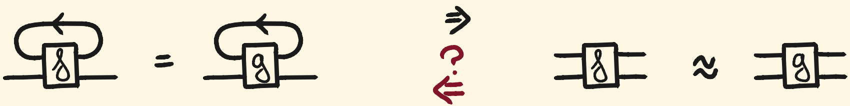
- formal compositional semantics for effectful stream computations
- trace equivalence and bisimulation of effectful Mealy machines
- characterisation as causal stream processes

FUTURE WORK

- coinduction up-to dinaturality



- Rel with explicit failure
- equality in St$\mathcal{C}$ implies bisimulation



- distance instead of equivalence relation for security