

EFFECTFUL MEALY MACHINES

BISIMULATION & TRACES

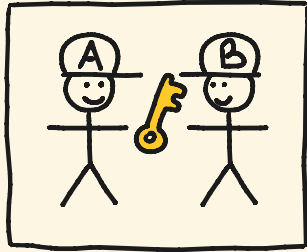
Silippo Bonchi
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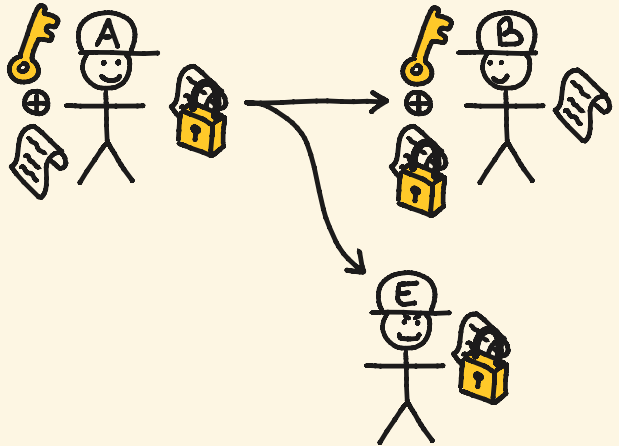
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ONE-TIME PAD PROTOCOL

1. share a key through a secure channel



2. send an encrypted message through a public channel



REPEATING THE ONE-TIME PAD

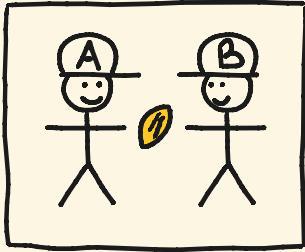
Sending n messages securely requires n private keys
↳ not very useful

- ⇒ • privately share a seed 🍀
- use identical pseudorandom number generator to obtain a new key for each message

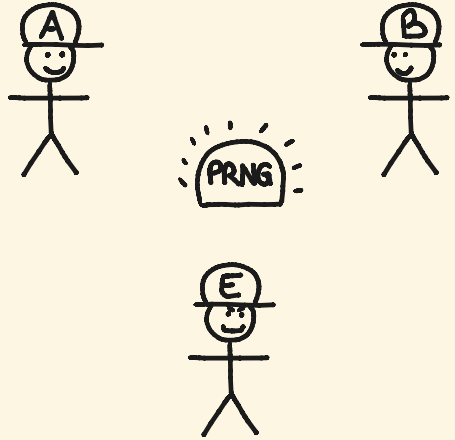


STREAM CIPHER PROTOCOL (1)

1. share a seed through a secure channel

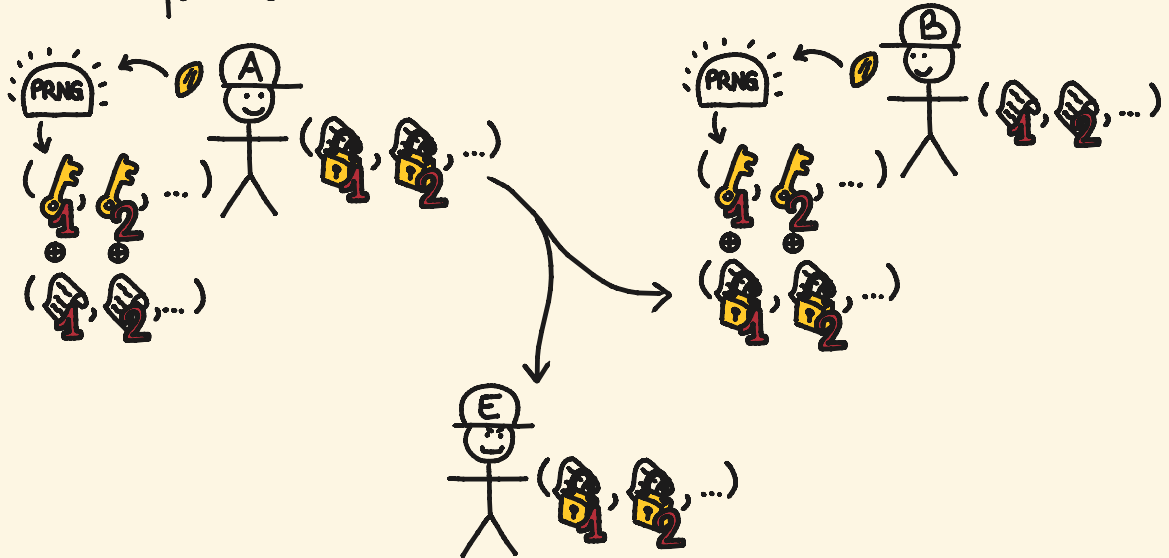


2. share a pseudorandom number generator



STREAM CIPHER PROTOCOL (2)

3. send a stream of encrypted messages through a public channel

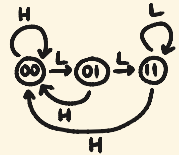


MEALY MACHINES

A classical Mealy machine is

$$\begin{cases} t: S \times A \rightarrow P(S \times B) & \text{transition relation} \\ s_0: 1 \rightarrow PS & \text{initial states} \end{cases}$$

TEMPERATURE
CONTROLLER



A Markov decision process is

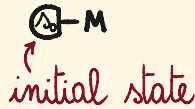
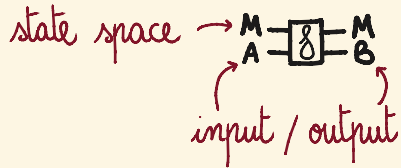
$$\begin{cases} t: S \times A \rightarrow \mathcal{D}S & \text{transition probability} \\ r: S \times A \rightarrow \mathcal{D}U & \text{reward} \\ s_0: 1 \rightarrow \mathcal{D}S & \text{initial distribution} \end{cases}$$

RANDOM
WALK



MEALY MACHINES

Systems are $f: M \otimes A \rightarrow M \otimes B$ with $s_0: I \rightarrow M$



- native sequential and parallel compositions
- parametric in the underlying process theory
- premonoidal categories for global effects

$\rightsquigarrow$ what is their behaviour?

when are two of them equivalent?

[cf. Katis, Sabadini, Walters 1997]

COALGEBRAIC SEMANTICS

Systems are coalgebras $f: M \rightarrow F(M)$

input/output

$$M \rightarrow (M \times B)^A$$

non-determinism

$$M \rightarrow \mathcal{P}(M \times B)$$

- bisimulation is equality in the final coalgebra

⇒ how do these compose?

how to change the underlying process theory?

MEALY MACHINES AND COALGEBRAS

Two faces of Mealy machines

$$M \times A \rightarrow \mathcal{D}(M \times B)$$

$$M \rightarrow \mathcal{D}(M \times B)^A$$

$$M \times A \rightarrow \mathcal{P}(M \times B)$$

$$M \rightarrow \mathcal{P}(M \times B)^A$$

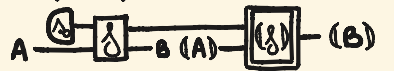


OVERVIEW

effectful Mealy machines



↓
bisimilarity

↓
effectful streams

[$\approx$ causal processes]

free construction
 $\sim$ syntax

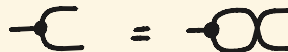
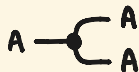
↓
uniformity

↓
coalgebraic construction
 $\sim$ trace semantics

OUTLINE

- effectful copy-discard categories
- effectful Mealy machines & bisimulations
- coinductive traces
- clausal traces

STRING DIAGRAMS



THE INTERCHANGE LAW

$$\begin{array}{c} A \text{ --- } \boxed{f} \text{ --- } B \\ A' \text{ --- } \boxed{g} \text{ --- } B' \end{array} = \begin{array}{c} A \text{ --- } \boxed{f} \text{ --- } B \\ A' \text{ --- } \boxed{g} \text{ --- } B' \end{array} = \begin{array}{c} A \text{ --- } \boxed{f} \text{ --- } B \\ A' \text{ --- } \boxed{g} \text{ --- } B' \end{array}$$

⇒ holds in monoidal categories

VALUES

Values are both :

- deterministic



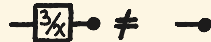
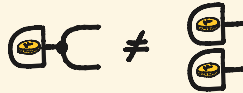
- total



ex $(3. -) : \mathbb{R} \rightarrow \mathbb{R}$

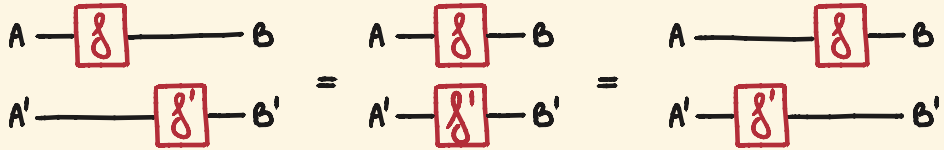
Non-ex $\text{Flip} : 1 \rightarrow \mathcal{D}(\{H, T\})$

$(3/-) : \mathbb{R} \rightarrow \mathbb{R}$



LOCAL COMPUTATIONS

local computations interchange,

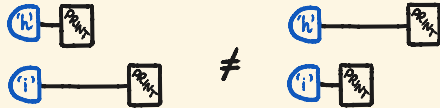


ex Stoch



EFFECTFUL COMPUTATIONS

Effectful computations may have global effects.

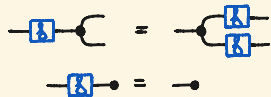


ex state monads, IO monad

[Pearce and Robinson (1997)]

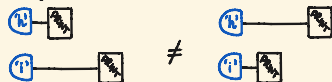
EFFECTFUL COPY-DISCARD CATEGORIES

Values can be
copied and discarded
(cartesian)

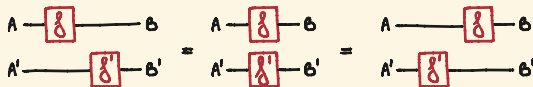


Effectful computations
may have global effects
(premonoidal)

$\mathcal{V} \rightarrow \mathcal{L} \rightarrow \mathcal{C}$



Local computations interchange
(monoidal)



[Jeffrey (1997), cf. Levy (2022), Power and Thielecke (1997)]

EXAMPLES OF EFFECTFUL TRIPLES

- $(\text{cart}(\mathcal{C}), \mathcal{Z}(\mathcal{C}), \mathcal{C})$ for a cs -premonoidal $\mathcal{C}$
- $(\text{Set}, \text{Kl}(\mathcal{Z}(T)), \text{Kl}(T))$ for a Set -monad T
- $(\text{Set}, \text{Set}, \text{Par})$
- $(\text{Set}, \text{Rel}_{\text{TOT}}, \text{Rel})$
- $(\text{Set}, \text{Stoch}, \text{Stoch}_{\leq})$
- $(\text{Set}, \text{Stoch}, \text{State}_S)$

STRING DIAGRAMS FOR PREMONOIDAL CATEGORIES

Add a "runtime" wire to non-central morphisms

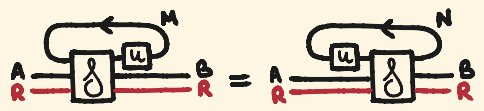
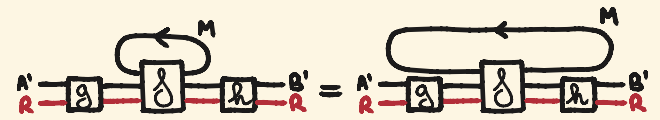
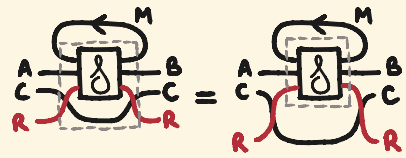
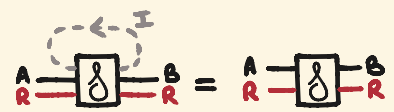
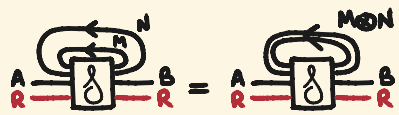
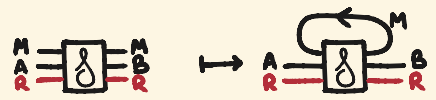
$$f: A \rightarrow B \quad \text{is} \quad \begin{array}{c} A \\ \text{---} \end{array} \boxed{f} \begin{array}{c} B \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ R \end{array}$$



[Jeffrey (1997); Román (2022); Román, Sobociński (2025)]

FEEDBACK EFFECTFUL CATEGORIES

$$\text{Fbk}_M: \mathcal{L}(M \otimes A, M \otimes B) \rightarrow \mathcal{L}(A, B)$$



$$u: N \rightarrow M \text{ in } \mathcal{L}$$



OUTLINE

- effectful copy-discard categories
- [• effectful Mealy machines & bisimulations]
- coinductive traces
- clausal traces

EFFECTFUL MEALY MACHINES

a Mealy machine $(f, M, s_0) : A \rightarrow B$ in $(\mathcal{V}, \mathcal{L}, \mathcal{C})$
is a morphism

$f : M \otimes A \rightarrow M \otimes B$ in $\mathcal{C}$ 
with an initial state
 $s_0 : I \rightarrow M$ in $\mathcal{L}$ 

ex $(\mathcal{Set}, \mathcal{Rel}_{\text{TOT}}, \mathcal{Rel})$

$$\begin{cases} f : M \times A \rightarrow \mathcal{P}(M \times B) \\ s_0 \in M \end{cases}$$

ex $(\mathcal{Set}, \mathcal{Stoch}, \mathcal{Stoch}_{\leftarrow})$

$$\begin{cases} f : M \times A \rightarrow \mathcal{D}(M \times B) \\ s_0 \in \mathcal{D}(M) \end{cases}$$

[cf. Katis, Sabadini, Walters (1997); EDL, Gianola, Román, Sabadini, Sobociński (2022)]

MORPHISMS OF MEALY MACHINES

A morphism of Mealy machines $u: (f, M, s_0) \rightarrow (g, N, t_0)$
is a value morphism $u: M \rightarrow N$ in $\mathcal{V}$

such that

$$\begin{array}{c} M \\ A \\ \hline \text{f} \\ \hline R \end{array} \begin{array}{c} \text{---} u \text{---} \\ \text{---} \end{array} \begin{array}{c} N \\ B \\ \hline \text{g} \\ \hline R \end{array} = \begin{array}{c} M \\ A \\ \hline \text{f} \\ \hline R \end{array} \begin{array}{c} \text{---} u \text{---} \\ \text{---} \end{array} \begin{array}{c} N \\ B \\ \hline \text{g} \\ \hline R \end{array}$$

$$\begin{array}{c} \text{f} \\ \hline \text{---} u \text{---} \\ \hline \end{array} \begin{array}{c} N \\ B \\ \hline \text{g} \\ \hline R \end{array} = \begin{array}{c} \text{f} \\ \hline \text{---} u \text{---} \\ \hline \end{array} \begin{array}{c} N \\ B \\ \hline \text{g} \\ \hline R \end{array}$$

ex (Set, Rel_{tot}, Rel)

$$\forall s \in M \forall a \in A \forall t \in N \forall b \in B$$

$$(t, b) \in g(u(s), a) \Leftrightarrow \exists s' \in M \ u(s') = t \wedge (s', b) \in f(s, a)$$

ex (Set, Stoch, Stoch_κ)

$$\forall s \in M \forall a \in A \forall t \in N \forall b \in B$$

$$g(t, b | u(s), a) = \sum_{s': u(s') = t} f(s', b | s, a)$$

EFFECTFUL CATEGORY OF MEALY MACHINES

Mealy is an effectful category where

- objects are the objects of $\mathcal{C}$
- morphisms $(f, M, s_0): A \rightarrow B$ are Mealy machines quotiented by **value** isomorphisms $u: M \xrightarrow{\cong} N$

$$\begin{array}{c} M \\ A \\ \hline R \end{array} \begin{array}{c} \text{f} \\ \hline \end{array} \begin{array}{c} \text{u} \\ \hline \end{array} \begin{array}{c} N \\ B \\ \hline R \end{array} = \begin{array}{c} M \\ A \\ \hline R \end{array} \begin{array}{c} \text{u} \\ \hline \end{array} \begin{array}{c} \text{g} \\ \hline \end{array} \begin{array}{c} N \\ B \\ \hline R \end{array}$$

$$\begin{array}{c} \text{f}_0 \\ \hline \end{array} \begin{array}{c} \text{u} \\ \hline \end{array} N = \begin{array}{c} \text{f}_0 \\ \hline \end{array} N$$

- composition tensors the state spaces

$\leadsto$ local states

$$\begin{array}{c} M \\ N \\ A \\ \hline R \end{array} \begin{array}{c} \text{f} \\ \hline \end{array} \begin{array}{c} M \\ N \\ C \\ \hline R \end{array} \begin{array}{c} \text{g} \\ \hline \end{array} \begin{array}{c} M \\ N \\ C \\ \hline R \end{array} = \begin{array}{c} \text{f}_0 \\ \hline \end{array} M \quad \begin{array}{c} \text{g}_0 \\ \hline \end{array} N$$

MEALY MACHINES ARE FREE

THEOREM

Mealy is the free pointed-feedback category over $\mathcal{C}$.

$$\text{Mealy}(A, B) = \int^{(\lambda_0, M) \in \text{pt} \mathcal{C}_{\text{iso}}} \mathcal{C}(M \otimes A, M \otimes B)$$



[cf. Katis, Sabadini, Walters (1997); EDL, Gianola, Román, Sabadini, Sobociński (2022)]

COALGEBRAIC MEALY MACHINES

$T: \text{Set} \rightarrow \text{Set}$ monad

$$F_T: \text{Set} \rightarrow \text{Set} \\ X \mapsto T(X \times B)^A$$

A T -Mealy machine is an F_T -coalgebra

$$f: X \rightarrow T(X \times B)^A$$

$$\Leftrightarrow \hat{f}: X \otimes A \rightarrow X \otimes B \quad \text{in } \text{Kl } T$$

$$\Leftrightarrow \text{effectful Mealy machine in} \\ (\text{Set}, \text{Kl}(\mathbb{Z}(T)), \text{Kl}(T))$$

COALGEBRAIC BISIMULATION

a bisimulation is a span of coalgebras.

$$\begin{array}{ccccc} M & \xleftarrow{\pi_1} & R & \xrightarrow{\pi_2} & N \\ \downarrow \delta & & \downarrow \alpha & & \downarrow \delta \\ F_T(M) & \xleftarrow{F_T(\pi_1)} & F_T(R) & \xrightarrow{F_T(\pi_2)} & F_T(N) \end{array}$$

THEOREM [Rutten (2000)]

When $F_T: \mathbf{Set} \rightarrow \mathbf{Set}$ preserves weak pullbacks,
bisimilarity is an equivalence relation.

[Aczel & Mendler (1989), Rutten (2000)]

BISIMULATION

For two effectful Mealy machines $(f, M, s), (g, N, t) : A \rightarrow B$, a bisimulation is a sequence of spans of morphisms.

$$(f, M, s) \xleftarrow{u_1} (h_1, R_1, r_1) \xrightarrow{v_1} (f_1, M_1, s_1) \xleftarrow{u_2} \dots (h_m, R_m, r_m) \xrightarrow{v_m} (g, N, t)$$

PROPOSITION

When $\mathcal{C} = \mathbf{Kl}(T)$, for a commutative monad T preserving weak pullbacks, effectful bisimulation coincides with coalgebraic bisimulation.

ex $\mathcal{C} = \mathbf{Set}, \mathbf{Par}, \mathbf{Rel}, \mathbf{Stoch}, \mathbf{pStoch}$

BISIMILARITY IS FREE

UNIFORM FEEDBACK [cf. Jasegarwa (2002); Lăzărescu, Stănescu (1994)]

$$\left\{ \begin{array}{l} \begin{array}{c} M \\ A \\ R \end{array} \begin{array}{c} \text{---} \text{---} \end{array} \begin{array}{c} \text{---} \text{---} \end{array} \begin{array}{c} \text{---} \text{---} \end{array} \begin{array}{c} N \\ B \\ R \end{array} \\ \text{---} \text{---} \end{array} = \begin{array}{c} M \\ A \\ R \end{array} \begin{array}{c} \text{---} \text{---} \end{array} \begin{array}{c} \text{---} \text{---} \end{array} \begin{array}{c} \text{---} \text{---} \end{array} \begin{array}{c} N \\ B \\ R \end{array} \\ \text{---} \text{---} \end{array} \right. \Rightarrow \begin{array}{c} A \\ R \end{array} \begin{array}{c} \text{---} \text{---} \end{array} \begin{array}{c} \text{---} \text{---} \end{array} \begin{array}{c} \text{---} \text{---} \end{array} \begin{array}{c} N \\ B \\ R \end{array} \\ \text{---} \text{---} \end{array} = \begin{array}{c} A \\ R \end{array} \begin{array}{c} \text{---} \text{---} \end{array} \begin{array}{c} \text{---} \text{---} \end{array} \begin{array}{c} \text{---} \text{---} \end{array} \begin{array}{c} N \\ B \\ R \end{array} \\ \text{---} \text{---} \end{array}$$

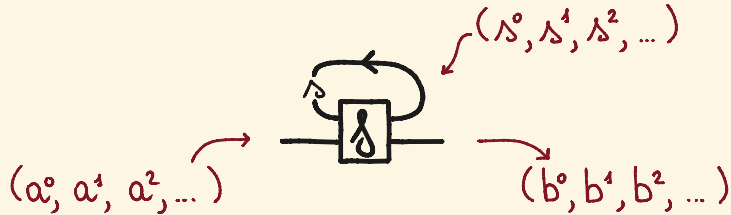
THEOREM

Mealy_{bisim} form the free uniform feedback structure over $(\mathcal{V}, \mathcal{L}, \mathcal{C})$.

OUTLINE

- effectful copy-discard categories
- effectful Mealy machines & bisimulations
- coinductive traces
- clausal traces

EXECUTING MEALY MACHINES



→ what should the semantic universe be?
when do two Mealy machines have the same executions?

STREAMS ARE COINDUCTIVE

A stream of elements of A is

- an element $a^0 \in A$
- a stream a^+ of elements of A

⇒ the set of streams is the final coalgebra of the functor

$$A \times (-) : \mathbf{Set} \rightarrow \mathbf{Set}$$

EFFECTFUL STREAMS

$A = (A^0, A^1, \dots)$, $B = (B^0, B^1, \dots)$ streams of objects of $(\mathcal{U}, \mathcal{L}, \mathcal{C})$.

An effectful stream $F: A \rightarrow B$ on $(\mathcal{U}, \mathcal{L}, \mathcal{C})$ is

- a memory $M_g \in \mathcal{L}$
- a first action $g^0: A^0 \rightarrow M_g \otimes B^0$ in $\mathcal{C}$
- the rest of the action $F^+: M_g \cdot A^+ \rightarrow B^+$

$$A \text{---} \boxed{F} \text{---} B = A^0 \text{---} \boxed{g^0} \text{---} \overset{M_g}{B^0 \text{---} A^+} \text{---} \boxed{F^+} \text{---} B^+$$

(Note: In the original image, the boxes for g^0 and F^+ are connected by a line labeled M_g above it, and the inputs/outputs are labeled with R and IR in red.)

quotiented by sliding: for $\pi: M_g \rightarrow M_g$ in $\mathcal{L}$

$$g^0 = g^0; (\pi \otimes 1)$$

$$g^+ = \pi \cdot F^+$$

$$\text{---} \boxed{g^0} \text{---} \boxed{F^+} \text{---} = \text{---} \boxed{g^0} \text{---} \boxed{\pi} \text{---} \boxed{F^+} \text{---} \sim \text{---} \boxed{g^0} \text{---} \boxed{\pi} \text{---} \boxed{F^+} \text{---} = \text{---} \boxed{g^0} \text{---} \boxed{g^+} \text{---}$$

(Note: In the original image, the boxes for π and g^+ are highlighted in red.)

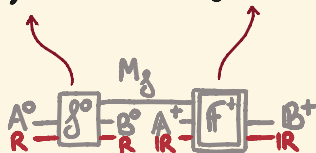
EFFECTFUL STREAMS

The profunctor Stream : $\mathcal{L}^{N^{\text{op}}} \times \mathcal{L}^N \rightarrow \text{Set}$ is the final coalgebra of the functor

$$F : [\mathcal{L}^{N^{\text{op}}} \times \mathcal{L}^N, \text{Set}] \rightarrow [\mathcal{L}^{N^{\text{op}}} \times \mathcal{L}^N, \text{Set}]$$

$$F(Q)(A, B) := \int^{M \in \mathcal{L}} \mathcal{L}(A^{\circ}, M \otimes B^{\circ}) \times Q(M \cdot A^{+}, B^{+})$$

quotient by sliding on the memory



COMPOSITIONAL STRUCTURE OF STREAMS

THEOREM

Effectful streams form an effectful category Stream .

- composition and monoidal actions are defined coinductively:
for $f: N_f \cdot A \rightarrow B$ and $g: N_g \cdot B \rightarrow C$,

$$\begin{cases} (f;_N g)^\circ := \text{Diagram 1} \\ (f;_N g)^+ := f^+;_M g^+ \end{cases}$$

Diagram 1: A box labeled f° with input $N_f \cdot A$ and output B is connected to a box labeled g° with input B and output $N_g \cdot C$. Red lines labeled R connect the output of f° to the input of g° .

$$\begin{cases} (X \otimes_N f)^\circ := \text{Diagram 2} \\ (X \otimes_N f)^+ := X^+ \otimes_M f^+ \end{cases}$$

Diagram 2: A box labeled f° with input $N_f \cdot A$ and output B is connected to a box labeled X° with input $N_X \cdot X$ and output $M_X \cdot X$. Red lines labeled R connect the output of f° to the input of X° .

FEEDBACK ON EFFECTFUL STREAMS

$\partial: \text{Stream} \rightarrow \text{Stream}$

$\partial(A) := (I, A^0, A^1, \dots)$

THEOREM

Stream has ∂ -feedback.

- feedback is defined coinductively

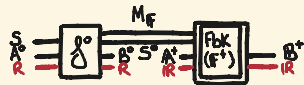
$$F: (S \cdot \partial S) \otimes A \rightarrow S \otimes B$$

$$\text{Fbk}_S F: S \cdot A \rightarrow B$$

$$M(\text{Fbk}_S^S F) := M(F) \otimes S^0$$

$$(\text{Fbk}_S^S F)^0 := \text{J}^0$$

$$(\text{Fbk}_S^S F)^+ := \text{Fbk}_{S^+}^S(F^+)$$

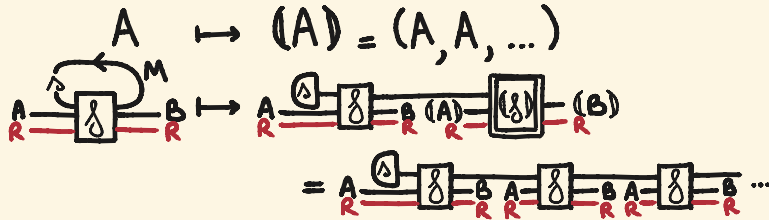


COMPOSITIONAL TRACE SEMANTICS

THEOREM

There is an effectful functor

$\text{Tr} : \text{Mealy} \rightarrow \text{Stream}$



TRACE IS UNIVERSAL

THEOREM

The trace functor $\text{Tr} : \text{Mealy} \rightarrow \text{Stream}$ is the unique feedback effectful functor determined by freeness of Mealy.

$$\begin{array}{ccc} & \text{FbEffCat} & \\ \text{Mealy} \begin{pmatrix} \uparrow \\ + \\ \downarrow \end{pmatrix} & \Rightarrow \exists! \text{Tr} : \text{Mealy}_{(V, Z, e)} \longrightarrow \text{Stream}_{(V, Z, e)} & \\ & \text{EffCat} & \\ & \text{Mealy}_{(V, Z, e)} \text{ / bisim} & \end{array}$$

COROLLARY

Bisimilarity implies trace equivalence.

OUTLINE

- effectful copy-discard categories
- effectful Mealy machines & bisimulations
- coinductive traces

[• clausal traces]

STREAM COMPUTATIONS

- Sliding equivalence might be difficult to handle
- Causal stream functions are old :

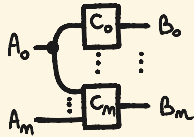
[Ramey (1958)] shows that they are
the executions of deterministic Mealy machines

⇒ is there a similar explicit form for effectful streams ?

CAUSAL STREAM FUNCTIONS

Stream computations $(c_m)_{m \in \mathbb{N}} : A \rightarrow B$ in a cartesian category are families $c_m : A_0 \times \dots \times A_m \rightarrow B_m$.

$\leadsto c_m(a_0, \dots, a_m) \in B_m$ is the output at time m



reconstructs the outputs until time m

[Sprunger & Katsumata (2019), Uustalu & Vene (2008)]

STOCHASTIC PROCESSES

Stochastic stream computations $(p_m)_{m \in \mathbb{N}} : A \rightarrow B$

are families $p_m : A_0 \times \dots \times A_m \longrightarrow \mathcal{D}(B_0 \times \dots \times B_m)$

such that $p_m(a_0, \dots, a_m) = \sum_{a \in A_{m+1}} p_{m+1}(a_0, \dots, a_m, a)$.

$\leadsto p_m(a_0, \dots, a_m) \in \mathcal{D}(B_0 \times \dots \times B_m)$ is the distribution of
the outputs until time n
 $\hookrightarrow$ the outputs may be correlated

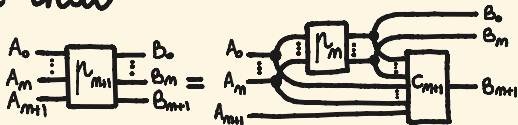
Is there a monoidal version of causal stream functions?

CAUSAL PROCESSES

A causal process $p: A \rightarrow B$ in a copy-discard category $\mathcal{C}$ is a family of morphisms

$$p_m: A_0 \otimes \dots \otimes A_m \rightarrow B_0 \otimes \dots \otimes B_m$$

such that



for some $c_{m+1}: B_0 \otimes \dots \otimes B_m \otimes A_0 \otimes \dots \otimes A_m \otimes A_{m+1} \rightarrow B_{m+1}$

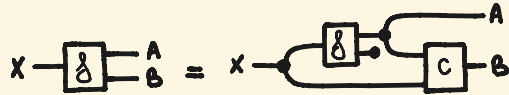
$\leadsto p_m$ determines the input-output behaviour
until time n

COMPOSING CAUSAL PROCESSES

$\mathcal{C}$ copy - discard

CONDITIONALS [Cho & Jacobs (2017), Fritz (2020), EDL & Román (2023)]

For all $f: X \rightarrow A \otimes B$ there is $c: A \otimes X \rightarrow B$ st



THEOREM

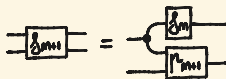
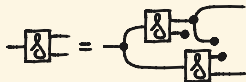
causal processes form a monoidal category Proc when $\mathcal{C}$ has conditionals with sharp domain.

CAUSAL PROCESSES : EXAMPLES

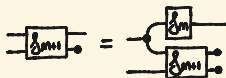
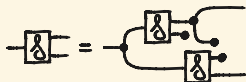
CONDITIONALS

CAUSALITY CONDITION

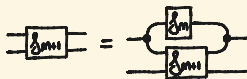
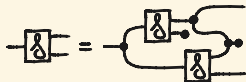
Set



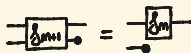
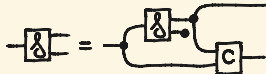
Par



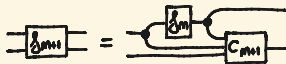
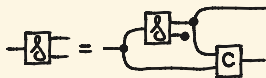
Rel



Stoch



pStoch



CAUSAL PROCESSES ARE STREAMS

ℓ copy - discard

RANGES

For all $m: X \rightarrow A$ there are $\begin{cases} \pi: A \otimes X \rightarrow R & \text{deterministic} \\ i: R \rightarrow A \otimes X & \text{total} \end{cases}$

$$x \rightarrow \boxed{m} \rightarrow A \quad = \quad x \rightarrow \boxed{m} \rightarrow \boxed{\pi} \rightarrow \boxed{i} \rightarrow A \otimes X$$

$$\begin{array}{c} x \\ y \end{array} \rightarrow \boxed{m} \rightarrow \boxed{c} \rightarrow B \quad = \quad \begin{array}{c} x \\ y \end{array} \rightarrow \boxed{m} \rightarrow \boxed{d} \rightarrow B \quad \Rightarrow \quad \begin{array}{c} R \\ y \end{array} \rightarrow \boxed{i} \rightarrow \boxed{c} \rightarrow B \quad = \quad \begin{array}{c} R \\ y \end{array} \rightarrow \boxed{i} \rightarrow \boxed{d} \rightarrow B$$

THEOREM

consider (func ℓ , tot ℓ , ℓ).

If ℓ has conditionals and ranges,
Proc $\approx$ Stream .

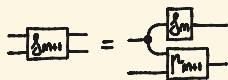
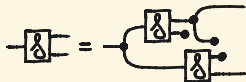
TRACES ARE EFFECTFUL TRACES

CONDITIONALS

CAUSALITY CONDITION

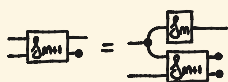
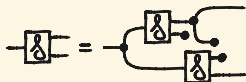
TRACE PREDICATE

Set



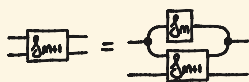
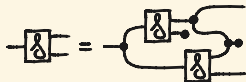
$$\mathcal{S}_0 = \mathcal{S} \wedge \forall i \leq n \\ (\mathcal{S}_{i+1}, b_i) = f(\mathcal{S}_i, a_i)$$

Par



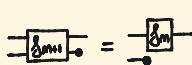
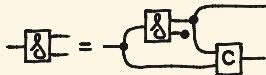
$$\mathcal{S}_0 = \mathcal{S} \wedge \forall i \leq n \\ (\mathcal{S}_{i+1}, b_i) = f(\mathcal{S}_i, a_i)$$

Rel



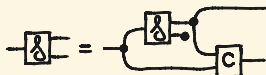
$$\exists \mathcal{S}_0, \dots, \mathcal{S}_{m+1} \mathcal{S}_0 \in \mathcal{S} \\ \wedge \forall i \leq n (\mathcal{S}_{i+1}, b_i) \in f(\mathcal{S}_i, a_i)$$

Stoch



$$\sum_{\mathcal{S}_0, \dots, \mathcal{S}_{m+1}} \mathcal{S}(\mathcal{S}_0) \\ \cdot \prod_{i \leq n} f(\mathcal{S}_{i+1}, b_i | \mathcal{S}_i, a_i)$$

pStoch



$$\sum_{\mathcal{S}_0, \dots, \mathcal{S}_{m+1}} \mathcal{S}(\mathcal{S}_0) \\ \cdot \prod_{i \leq n} f(\mathcal{S}_{i+1}, b_i | \mathcal{S}_i, a_i)$$

SUMMARY

effectful Mealy machines



↓
bisimilarity

↓
effectful streams

[$\approx$ causal processes]

free construction
 $\sim$ syntax

↓
uniformity

↓
coalgebraic construction
 $\sim$ trace semantics

FUTURE WORK

- Adding choice, iteration and higher-order
- Behavioural metrics, metric enrichment
- Coinduction up-to dinaturality
- Tree-like behaviour via final coalgebra of an appropriate ψ

$$\boxed{g^0} - \boxed{f^+} = \boxed{g^0} - \boxed{\pi} - \boxed{f^+} \sim \boxed{g^0} - \boxed{\pi} - \boxed{f^+} = \boxed{g^0} - \boxed{g^+}$$

[Filippo Bonchi, EDL, Mario Román

Effectful Mealy machines: bisimulation and trace (2025) LiCS]