Stala

23 December 2025

PROGRAM LOGICS VIA DISTRIBUTIVE MONOIDAL CATEGORIES

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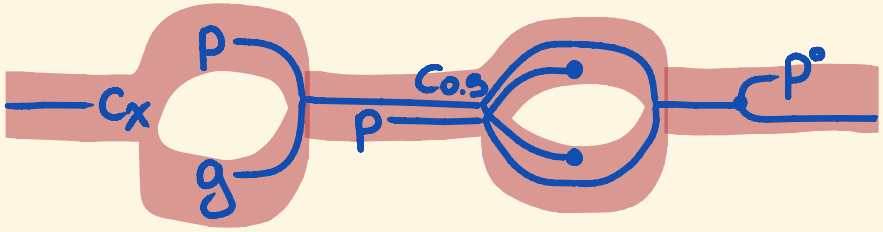
THE PIRAÑHA PROBLEM

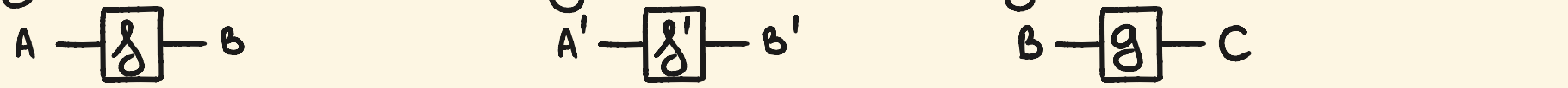
```
if coin(x)
| then F1 ← piranha
| else F1 ← goldFish
end
F2 ← piranha
if coin(0.5)
| then s ← F1
| else s ← F2
end
observe(s = piranha)
```



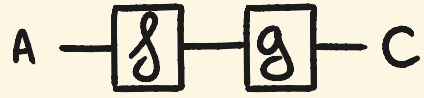
⇝ can we bound the probability of observing piranha ?

MORPHISMS AS PROGRAMS

$f : A \to B$ $\quad f' : A' \to B'$ $\quad g : B \to C$

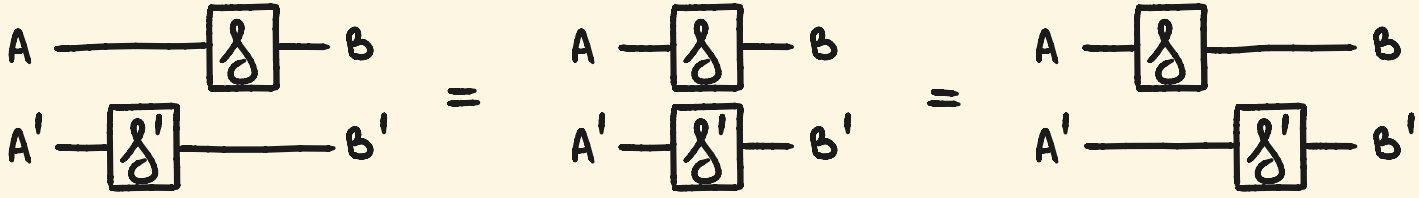


- Sequential composition. $f ; g : A \to C$



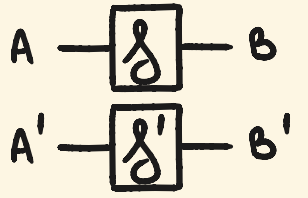
- Identity. $id_A : A \to A$

- Parallel composition. $f \otimes f' : A \otimes A' \to B \otimes B'$



- Symmetries. $\sigma_{A,B} : A \otimes B \to B \otimes A$

TWO KINDS OF PARALLEL COMPOSITION



execute f and f' ⇝ data flow

execute f or f' ⇝ control flow

ex for $T : \mathrm{Set} \to \mathrm{Set}$ x-monoidal monad,

$(\mathrm{Kl}T, \otimes, \{*\})$ ⇝ 'and'

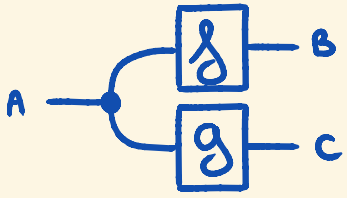
$(\mathrm{Kl}T, +, \emptyset)$ ⇝ 'or'

We need both kinds of parallel compositions

⇒ distributive monoidal categories $(\mathcal{C}, \otimes, 1, +, 0)$

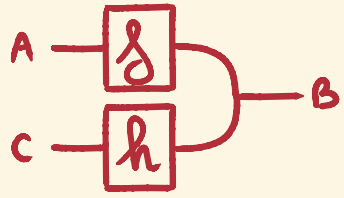
TWO MONOIDAL STRUCTURES

data flow



$A \longrightarrow B \otimes C$

control flow



$A + C \longrightarrow B$

[Arbib, Manes (1980)] [Walters (1992)]
[Jacobs (2010, 2016)]

OUTLINE

- [The guarded command language]
- Imperative categories
- Program logics

THE GUARDED COMMAND LANGUAGE (GCL)

- Signature

  $Var, \Sigma_E, \Sigma_G, \Sigma_P, \Sigma_C, \Sigma_D$

- Expressions

  $E ::= x \in Var \mid e(E, \ldots, E),\ e \in \Sigma_E$

- Guards

  $G ::= b(E, \ldots, E),\ b \in \Sigma_G \mid \bot \mid \top \mid G \wedge G \mid \neg G$

- Predicates

  $P ::= p(E, \ldots, E),\ p \in \Sigma_P \mid \top \mid \bot \mid P \wedge P \mid P +_G P \mid \check{G}$

- Commands

  $C ::= c \in \Sigma_C \mid skip \mid abort \mid x \leftarrow E \mid (x_1, \ldots, x_m) \xleftarrow{\$} \sigma,\ \sigma \in \Sigma_D \mid C;C$
  $\mid assert\ P \mid if\ B\ then\ C\ else\ C \mid while\ B\ do\ C$

INTERPRETING THE GUARDED COMMAND LANGUAGE

- Choose a type X for variables and fix $N := \#\mathrm{Var}$.
- Commands are endomorphisms of $X^N = \underbrace{X \otimes \cdots \otimes X}_{N \text{ times}}$
  ⇝ state-passing translation
- Expressions are deterministic and total $X^N \longrightarrow X$
- Guards are total $X^N \longrightarrow 1+1$
- Predicates are $X^N \longrightarrow 1$

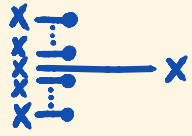
→ we also constructed an internal language (in the paper)
  ↳ succint with few primitives
  ↳ programs with arbitrary types

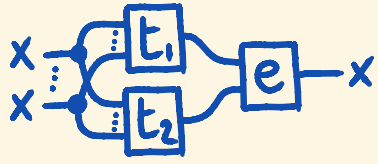
INTERPRETING THE GCL : EXPRESSIONS

$E ::= x_i \in Var \mid e(E, \dots, E),\ e \in \Sigma_E(n)$

Choose $\bar{e} : X^n \longrightarrow X$ for each $e \in \Sigma_E(n)$.

Define

$[\![x_i]\!] := \pi_i =$ 

$[\![e(t_1, \dots, t_m)]\!] := X^N \to m \,;\, ([\![t_1]\!] \otimes \cdots \otimes [\![t_m]\!]) \,;\, \bar{e} \overset{m=2}{=}$ 

⇒ copy-discard structure for variable handling
↳ compatible cocommutative comonoid on every object

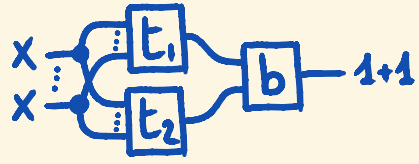
INTERPRETING THE GCL : GUARDS

$G ::= b(E,\dots,E),\ b \in \Sigma_G(n) \mid L \mid R \mid G \wedge G \mid \neg G$

Choose $\mathfrak{b} : X^n \longrightarrow 1+1$ total for each $b \in \Sigma_G(n)$

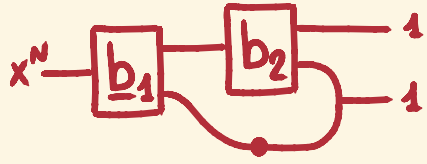
Define

$[\![b(t_1,\dots,t_m)]\!] := X^N \to X^m ; ([\![t_1]\!] \otimes \dots \otimes [\![t_m]\!]) ; \mathfrak{b}$ $\overset{m=2}{=}$



$[\![L]\!] := X^N \to ; \vdash 1,1 = X^N \to \vdash 1,1$

$[\![b_1 \wedge b_2]\!] :=$



$[\![R]\!] := X^N \to ; \dashv 1,1 = X^N \dashv 1,1$

$[\![\neg b]\!] :=$

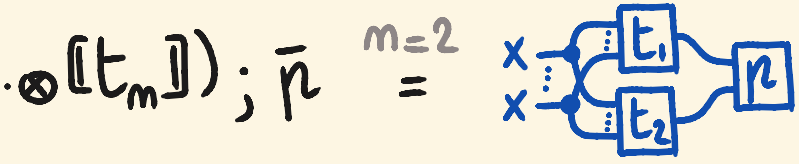
$\Rightarrow$ coproducts for choices

INTERPRETING THE GCL : PREDICATES

$P ::= p(E,\dots,E),\ p \in \Sigma_P(n) \mid \top \mid \bot \mid P \wedge P \mid P +_G P \mid \check{G}$

Choose $\bar{p} : X^n \longrightarrow 1$ for each $p \in \Sigma_P(n)$

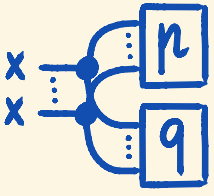
Define

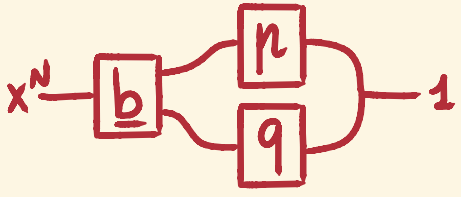
$[\![p(t_1,\dots,t_m)]\!] :=$ 

$[\![\top]\!] :=$ 

$[\![\bot]\!] :=$ 

$[\![\check{b}]\!] :=$ 

$[\![p \wedge q]\!] :=$ 

$[\![p +_b q]\!] :=$ 

INTERPRETING THE GCL : COMMANDS (1)

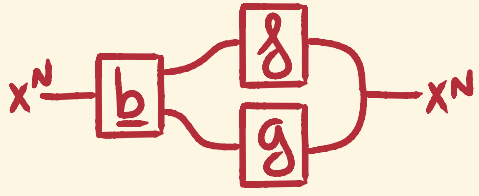
$C ::= c \in \Sigma_C \mid \text{skip} \mid \text{abort} \mid x \leftarrow E \mid (x_1, \ldots, x_m) \overset{\$}{\leftarrow} \sigma, \sigma \in \Sigma_D \mid C;C$
$\mid \text{assert } P \mid \text{if } B \text{ then } C \text{ else } C \mid \text{while } B \text{ do } C$

Choose $\bar{c} : X^N \to X^N$ for each $c \in \Sigma_C$
and $\bar{\sigma} : 1 \to X^m$ for each $\sigma \in \Sigma_D$

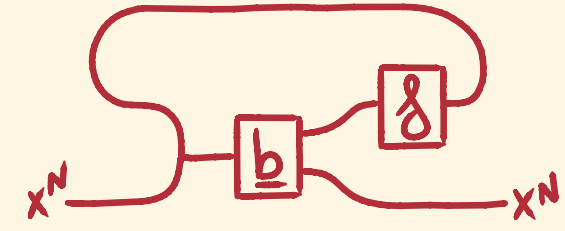
Define

$[\![\text{abort}]\!] := X^N \dashv\vdash X^N$

$[\![\text{if } b \text{ then } f \text{ else } g]\!]$
$:= [\![\underline{b}]\!] ; [[\![f]\!], [\![g]\!]]$



$[\![\text{while } b \text{ do } f]\!]$
$:= tr_{X^N}(\supset X^N ; [\![\underline{b}]\!] ; ([\![f]\!] + id_{X^N}))$



$\Rightarrow$ Traces for while loops

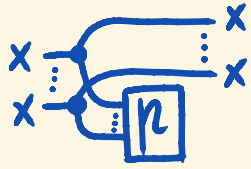
INTERPRETING THE GCL : COMMANDS (2)

$C ::= c \in \Sigma_C \mid \mathsf{skip} \mid \mathsf{abort} \mid x \leftarrow E \mid (x_1, \ldots, x_m) \xleftarrow{\$} \sigma,\ \sigma \in \Sigma_D \mid C;C$

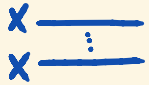
$\mid \mathsf{assert}\ P \mid \mathsf{if}\ B\ \mathsf{then}\ C\ \mathsf{else}\ C \mid \mathsf{while}\ B\ \mathsf{do}\ C$

Define

$[\![\mathsf{assert}\ p]\!] := X^N \to C ; (id_{X^N} \otimes [\![p]\!])$ =



$[\![\mathsf{skip}]\!] := id_{X^N}$ =

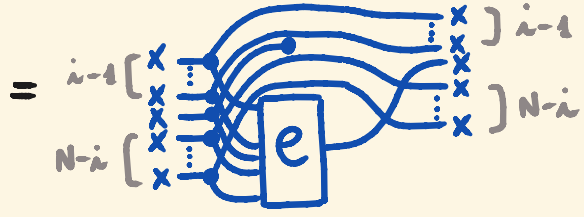


$[\![g;g]\!] := [\![g]\!]; [\![g]\!]$ =

$[\![x_i \leftarrow e]\!]$

$:= X^N \to C ; (\pi_{-i} \otimes [\![e]\!]) ; \chi_{X^{N-i}}^{X^{i-1}}$

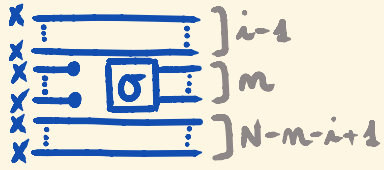
=



$[\![(x_i, \ldots, x_{i+m}) \xleftarrow{\$} \sigma]\!]$

$:= id_{X^{i-1}} \otimes (X^m \to \bullet ; \sigma - X^m) \otimes id_{X^{N-m-i+1}}$

=



OUTLINE

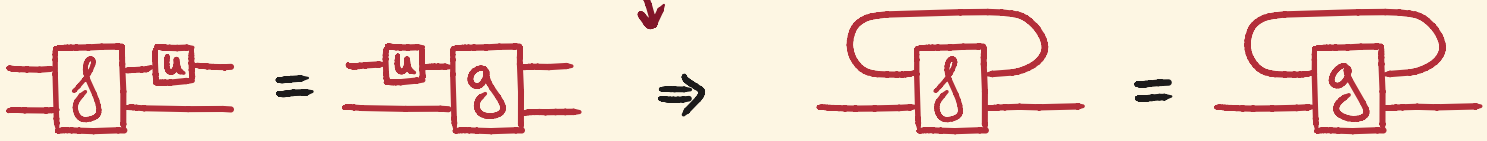
- The guarded command language
- [Imperative categories]
- Program logics

IMPERATIVE CATEGORIES

Category $\mathcal{C}$ with

- a copy-discard structure $(\otimes, 1)$
- finite coproducts $(+, 0)$
- uniform coproduct-trace $tr_S : \mathcal{C}(S+A, S+B) \to \mathcal{C}(A, B)$

$\mapsto$ distributive cd (injections det & tot)



$(\mathcal{C}, \otimes, I)$ symmetric monoidal category with coproducts and

$$\delta^L_{A,B,C} : (A \otimes B) + (A \otimes C) \xrightarrow{\simeq} A \otimes (B + C) \qquad \jmath^L_A : 0 \xrightarrow{\simeq} 0 \otimes A$$

$$\delta^R_{A,B,C} : (A \otimes C) + (B \otimes C) \xrightarrow{\simeq} (A + B) \otimes C \qquad \jmath^R_A : 0 \xrightarrow{\simeq} A \otimes 0$$

UNIFORMITY GIVES DIVERGENCE

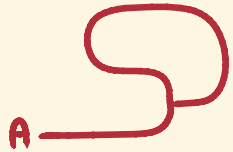
PROPOSITION

$\mathcal{C}$ category with coproducts and uniform trace

$\Rightarrow$ *0 is terminal*

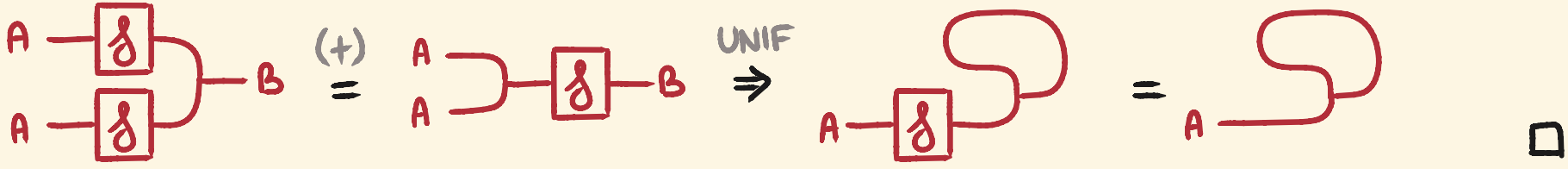
PROOF

Consider the morphisms $A\dashv : A \longrightarrow 0$ defined

as $A\dashv := tr_A[id_A, id_A] =$ 

$\rightsquigarrow$ while TRUE do skip

By uniformity, we show that they are natural.



□

EXAMPLES

Kleisli categories of partially additive monads

- $\mathbf{Par}$, sets and partial functions
- $\mathbf{Rel}$, sets and relations
- $\mathbf{Stoch}_{\leq}$, sets and countably supported partial stochastic kernels
- $\mathbf{BorelStoch}_{\leq}$, standard Borel spaces and measurable partial stochastic kernels

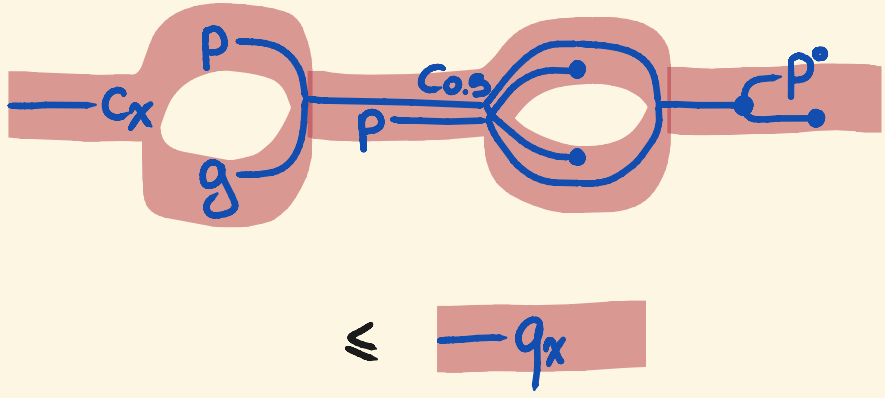
[Jacobs (2010), Jlasuo (2006)]

OUTLINE

- The guarded command language
- Imperative categories
- [Program logics]

THE PIRAÑHA PROBLEM

```
if coin(x)
| then F₁ ← piranha
| else F₁ ← goldFish
end
F₂ ← piranha
if coin(0.5)
| then s ← F₁
| else s ← F₂
end
observe(s = piranha)
```

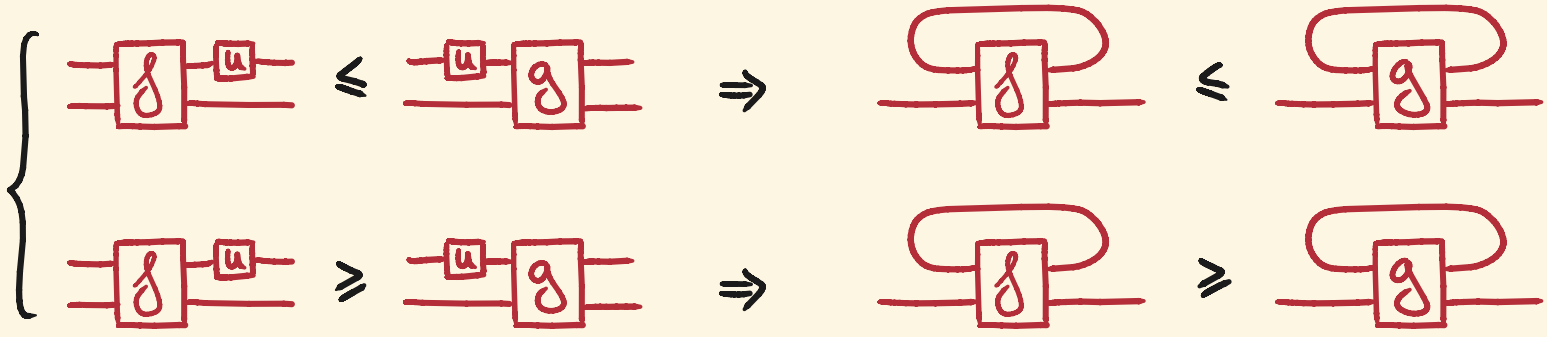


⇝ can we bound the probability of observing piranha ?
↳ we need an order on morphisms

POSETAL IMPERATIVE CATEGORY

A imperative category with

- a poset distributive enrichment
- posetally-uniform coproduct-trace



EXAMPLES

All our examples are poset-enriched.

- $\mathbf{Par}$, $f \leq g$ if $\forall a \in \mathrm{dom}\, f \quad f(a) = g(a)$
- $\mathbf{Rel}$, $f \leq g$ if $f \subseteq g$ as relations
- $\mathbf{Stoch}_{\leq}$, $f \leq g$ if $\forall a \in A \; \forall b \in B \quad f(b|a) \leq g(b|a)$
- $\mathbf{BorelStoch}_{\leq}$, $f \leq g$ if $\forall a \in A \; \forall E \in \sigma_B \quad f(E|a) \leq g(E|a)$

PROGRAM TRIPLES

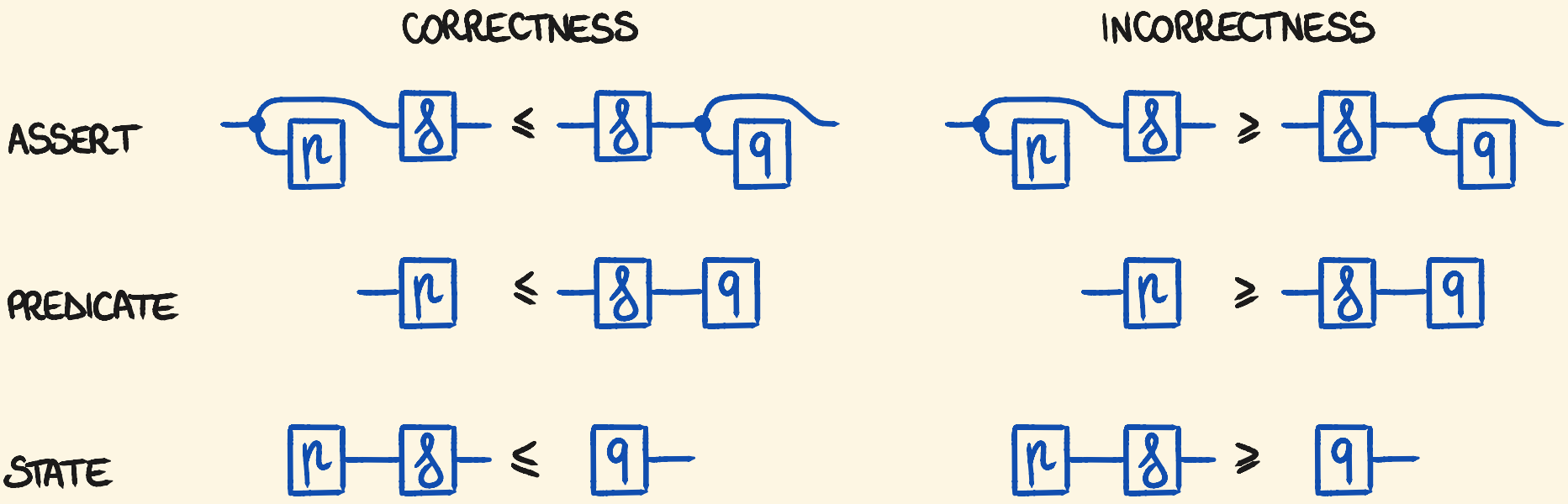
Statements of the form $\{p\}\ S\ \{q\}$.

program

precondition

postcondition

When is a triple valid?



HOARE LOGIC (ASSERT-CORRECTNESS) [Hoare (1969)]

$$\frac{}{\{p\}\ \mathrm{skip}\ \{p\}} \qquad \frac{\{p\}\ f\ \{q\} \quad \{q\}\ g\ \{r\}}{\{p\}\ f;g\ \{r\}} \qquad \frac{p \leqslant p' \quad \{p'\}\ f\ \{q'\} \quad q' \leqslant q}{\{p\}\ f\ \{q\}} \qquad \frac{\{p_1\}\ f\ \{q_1\} \quad \{p_2\}\ f\ \{q_2\}}{\{p_1 \wedge p_2\}\ f\ \{q_1 \wedge q_2\}}$$

$$\frac{}{\{p\}\ \mathrm{abort}\ \{q\}} \qquad \frac{}{\{p(x_1,\dots,x_{i-1},e,x_{i+1},\dots,x_N)\}\ x_i \leftarrow e\ \{p\}} \qquad \frac{}{\{q\}\ \mathrm{assert}\ p\ \{p \wedge q\}} \qquad \frac{q \wedge r \leqslant \bot}{\{r +_b q\}\ \mathrm{assert}\ r\ \{p \wedge \overline{b}\}}$$

$$\frac{}{\{\bot\}\ f\ \{q\}} \qquad \frac{}{\{p\}\ f\ \{\top\}} \qquad \frac{\{p\}\ f\ \{q\} \quad \{p\}\ g\ \{q\}}{\{p\}\ \mathrm{if}\ b\ \mathrm{then}\ f\ \mathrm{else}\ g\ \{q\}} \qquad \frac{\{p \wedge b\}\ f\ \{q\} \quad \{p \wedge \neg b\}\ g\ \{q\} \quad b\ \mathrm{det}}{\{p\}\ \mathrm{if}\ b\ \mathrm{then}\ f\ \mathrm{else}\ g\ \{q\}}$$

$$\frac{\{p\}\ \mathrm{if}\ b\ \mathrm{then}\ (f;\mathrm{while}\ b\ \mathrm{do}\ f)\ \mathrm{else}\ \mathrm{skip}\ \{q\}}{\{p\}\ \mathrm{while}\ b\ \mathrm{do}\ f\ \{q\}} \qquad \frac{\{p\}\ f\ \{p\}}{\{p\}\ \mathrm{while}\ b\ \mathrm{do}\ f\ \{p\}} \qquad \frac{\{p \wedge b\}\ f\ \{p\} \quad b\ \mathrm{det}}{\{p\}\ \mathrm{while}\ b\ \mathrm{do}\ f\ \{p \wedge \neg b\}}$$

THEOREM

The rules of assert-correctness program logics are valid in any posetal imperative category where, for all morphisms f and all predicates p, $\mathrm{abort} \leqslant f$ and $p \leqslant \top$.

INCORRECTNESS LOGIC (STATE-INCORRECTNESS)

[de Vries, Koutavas (2011), O'Hearn (2019)]

$$\frac{}{\{s\}\ \mathsf{skip}\ \{s\}} \qquad \frac{\{s\}\ f\ \{t\} \quad \{t\}\ g\ \{u\}}{\{s\}\ f;g\ \{u\}} \qquad \frac{\{s\}\ f\ \{\bot\}}{\{s\}\ f;g\ \{\bot\}} \qquad \frac{s \geqslant s' \quad \{s'\}\ f\ \{t'\} \quad t' \geqslant t}{\{s\}\ f\ \{t\}} \qquad \frac{\{s_1\}\ f\ \{t_1\} \quad \{s_2\}\ f\ \{t_2\} \quad b\ \text{const}}{\{s_1 +_b s_2\}\ f\ \{t_1 +_b t_2\}}$$

$$\frac{}{\{s\}\ x_i \leftarrow e\ \{s[x_i \backslash e]\}} \qquad \frac{}{\{s\}\ (x_{i_1}, \ldots, x_{i_m}) \xleftarrow{\$} s_0\ \{\coprod_{x_i} s \cdot s_0\}} \qquad \frac{}{\{s\}\ \mathsf{abort}\ \{\bot\}} \qquad \frac{}{\{s\}\ f\ \{\bot\}} \qquad \frac{}{\{s\}\ \mathsf{assert}\ p\ \{s \downarrow p\}}$$

$$\frac{\{s \downarrow b\}\ f\ \{t\}}{\{s\}\ \mathsf{if}\ b\ \mathsf{then}\ f\ \mathsf{else}\ g\ \{t\}} \qquad \frac{\{s \downarrow \neg b\}\ g\ \{t\}}{\{s\}\ \mathsf{if}\ b\ \mathsf{then}\ f\ \mathsf{else}\ g\ \{t\}} \qquad \frac{\{s \downarrow b\}\ f; \mathsf{while}\ b\ \mathsf{do}\ f\ \{t\}}{\{s\}\ \mathsf{while}\ b\ \mathsf{do}\ f\ \{t\}} \qquad \frac{}{\{s\}\ \mathsf{while}\ b\ \mathsf{do}\ f\ \{s \downarrow \neg b\}}$$

THEOREM

The rules of state-incorrectness program logics are valid in any positive imperative category where, for all morphisms f, $\mathsf{abort} \leqslant f$.

OUTCOME LOGIC (PREDICATE-CORRECTNESS)

[Zilberstein, Dreyer, Silva (2023)]

$$\frac{}{\{p\}\ \mathsf{skip}\ \{p\}} \qquad \frac{\{p\}\,f\,\{q\} \quad \{q\}\,g\,\{r\}}{\{p\}\,f;g\,\{r\}} \qquad \frac{p \leqslant p' \quad \{p'\}\,f\,\{q'\} \quad q' \leqslant q}{\{p\}\,f\,\{q\}} \qquad \frac{\{p_1\}\,f\,\{q_1\} \quad \{p_2\}\,f\,\{q_2\} \quad b\ \text{const}}{\{p_1 +_b p_2\}\,f\,\{q_1 +_b q_2\}} \qquad \frac{\neg b \wedge q \leqslant \bot \quad b\ \text{det}}{\{p +_b q\}\ \mathsf{assert}\ b\ \{p\}}$$

$$\frac{}{\{p(x_1,\ldots,x_{i-1},e,x_{i+1},\ldots,x_N)\}\ x_i \leftarrow e\ \{p\}} \qquad \frac{}{\{p(x_i,\ldots,x_{i+m} \setminus s)\}\ x_i,\ldots,x_{i+m} \xleftarrow{\$} s\ \{p\}}$$

$$\frac{\{p\}\ \mathsf{if}\ b\ \mathsf{then}\ (f;\ \mathsf{while}\ b\ \mathsf{do}\ f)\ \mathsf{else}\ \mathsf{skip}\ \{q\}}{\{p\}\ \mathsf{while}\ b\ \mathsf{do}\ f\ \{q\}}$$

$$\frac{\{p \wedge b\}\,f\,\{q\} \quad \{p \wedge \neg b\}\,g\,\{q\} \quad b\ \text{det}}{\{p\}\ \mathsf{if}\ b\ \mathsf{then}\ f\ \mathsf{else}\ g\ \{q\}} \qquad \frac{\{p\}\,f\,\{q\} \quad \{p\}\,g\,\{q\}}{\{p\}\ \mathsf{if}\ b\ \mathsf{then}\ f\ \mathsf{else}\ g\ \{q\}} \qquad \frac{}{\{\bot\}\,f\,\{q\}} \qquad \frac{}{\{p \wedge q\}\ \mathsf{assert}\ p\ \{q\}}$$

THEOREM

The rules of predicate-correctness program logics are valid in any posetal imperative category where, for all morphisms f, $\mathsf{abort} \leqslant f$.

RELATIONAL PROGRAM TRIPLES

Statements of the form $\{p\}\ f \sim g\ \{q\}$.

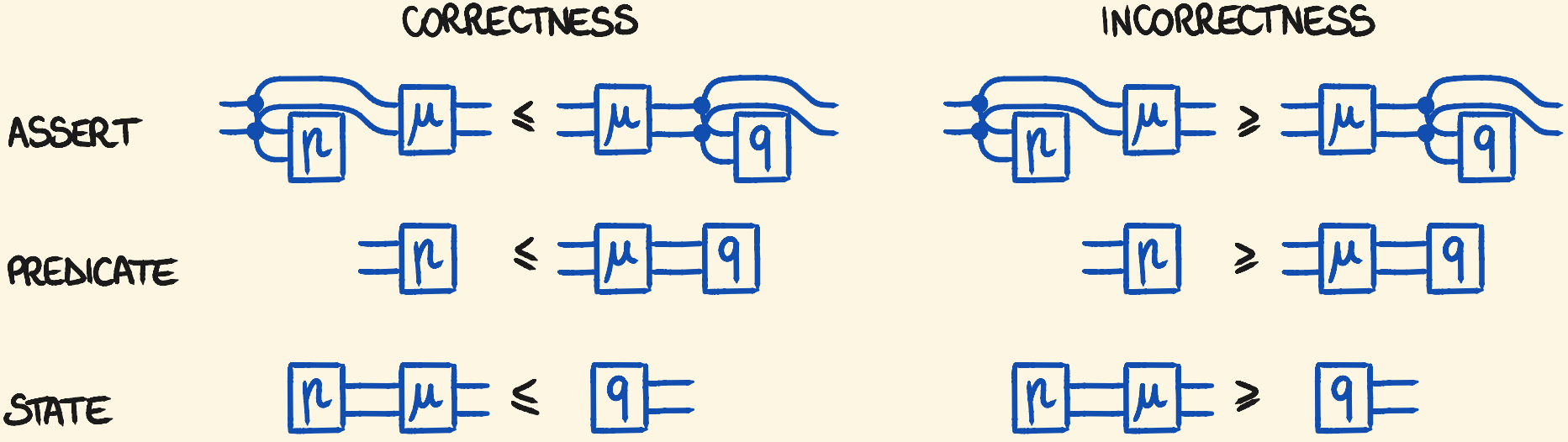
programs

precondition

postcondition

When is a triple valid?

There is a coupling $\mu \triangleright f \& g$ such that



[Benton (2004)] [Avanzini, Barthe, Davoli, Lyrégoire (2025)]

COUPLINGS

A strict coupling μ of $f : A \to A$ and $g : B \to B$ is $\mu : A \otimes B \to A \otimes B$ st

$\rightsquigarrow$ this forces f and g to have the same termination behaviour

A coupling μ of $f : A \to A$ and $g : B \to B$ is $\mu : A \otimes B \to A \otimes B + A + B$ st

RELATIONAL HOARE LOGIC (ASSERT-CORRECTNESS)

$$\frac{}{\{p\}\ \mathsf{skip} \sim \mathsf{skip}\ \{p\}} \qquad \frac{\{p\}\ c_1 \sim d_1\ \{q\} \quad \{q\}\ c_2 \sim d_2\ \{r\}}{\{p\}\ c_1 c_2 \sim d_1 d_2\ \{r\}} \qquad \frac{p \leqslant p' \quad \{p'\}\ c \sim d\ \{q'\} \quad q' \leqslant q}{\{p\}\ c \sim d\ \{q\}} \qquad \frac{\{p\}\ c \sim d\ \{q\}}{\{\sigma p\}\ c \sim d\ \{\sigma q\}}$$

$$\frac{}{\{p[(u_1,u_2)\backslash(e_1,e_2)]\}\ u_1 \leftarrow e_1 \sim u_2 \leftarrow e_2\ \{p\}} \qquad \frac{\{p\}\ c_1 \sim c_2\ \{q\} \quad \{p\}\ c_1 \sim d_2\ \{q\} \quad \{p\}\ d_1 \sim c_2\ \{q\} \quad \{p\}\ d_1 \sim d_2\ \{q\}}{\{p\}\ \mathsf{if}\ b_1\ \mathsf{then}\ c_1\ \mathsf{else}\ d_1 \sim \mathsf{if}\ b_2\ \mathsf{then}\ c_2\ \mathsf{else}\ d_2\ \{q\}}$$

$$\frac{\{(\check{b}_1 \otimes \check{b}_2) \wedge p\}\ c_1 \sim c_2\ \{q\} \quad \{(\neg\check{b}_1 \otimes \neg\check{b}_2) \wedge p\}\ d_1 \sim d_2\ \{q\} \quad b_1, b_2\ \mathsf{det}}{\{(b_1 = b_2) \wedge p\}\ \mathsf{if}\ b_1\ \mathsf{then}\ c_1\ \mathsf{else}\ d_1 \sim \mathsf{if}\ b_2\ \mathsf{then}\ c_2\ \mathsf{else}\ d_2\ \{q\}}$$

$$\frac{\{(\check{b}_1 \otimes \check{b}_2) \wedge p\}\ c_1 \sim c_2\ \{(b_1 = b_2) \wedge p\} \quad b_1, b_2\ \mathsf{det}}{\{(b_1 = b_2) \wedge p\}\ \mathsf{while}\ b_1\ \mathsf{do}\ c_1 \sim \mathsf{while}\ b_2\ \mathsf{do}\ c_2\ \{(\neg\check{b}_1 \otimes \neg\check{b}_2) \wedge p\}} \qquad \frac{\{p\}\ c_1 \sim c_2\ \{p\} \quad \{p\}\ c_1 \sim \mathsf{skip}\ \{p\} \quad \{p\}\ \mathsf{skip} \sim c_2\ \{p\}}{\{p\}\ \mathsf{while}\ b_1\ \mathsf{do}\ c_1 \sim \mathsf{while}\ b_2\ \mathsf{do}\ c_2\ \{p\}}$$

+ one-sided rules

THEOREM

The rules of relational assert-correctness program logics are valid in any posetal imperative category where, for all morphisms f, $\mathsf{abort} \leqslant f$.

SUMMARY

- Imperative categories are a good semantic universe for imperative programs (with effects).
- Poretal imperative categories derive program logics, existing and new
- Sound and complete syntax (in the paper)

FUTURE WORK

- Separation logic & premonoidality
- Predicate fibrations for richer predicates
- Relative completeness