SYCO 8 - Tallinn

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DIALECTICA PETRI NETS

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MOTIVATION & INTRODUCTION

- combine Petri nets using linear logic connectives
- Dialectica construction [1]
  - → models of linear logic
  - → category of Petri nets [2]

[1] V. de Paiva, The Dialectica categories, PhD thesis 1991
[2] C. Brown and D. Gurr, A categorical linear framework for Petri nets, 1995

OUTLINE

- PART 0 : the Dialectica construction
- PART 1 : linear logic structure
- PART 2 : changing the arcs

DIALECTICA CONSTRUCTION

LINEALE

$(L, *, e, \multimap, \leq)$ is a monoidal closed poset

$\rightsquigarrow \begin{cases} a \leq a' \\ b \leq b' \end{cases} \Rightarrow \begin{cases} a * b \leq a' * b' \\ a' \multimap b \leq a \multimap b' \end{cases}$ and $b * c \leq a \Leftrightarrow b \leq c \multimap a$

$(* \dashv \multimap)$

DIALECTICA CATEGORY Dial_L

- objects are (U, X, α) with $\alpha : \overset{\text{transitions}}{U} \times \overset{\text{places}}{X} \longrightarrow L$ in $\mathcal{S}et$

  $\rightsquigarrow$ 'L-valued relations'

- morphisms are $(f, F) : (U, X, \alpha) \longrightarrow (V, Y, \beta)$ with

$\begin{cases} f : U \longrightarrow V \\ F : Y \longrightarrow X \end{cases}$ such that

$$\begin{tikzcd}
U \times Y \arrow[r, "f \times 1"] \arrow[d, "1 \times F"'] \arrow[dr, phantom, "\leq"] & V \times Y \arrow[d, "\beta"] \\
U \times X \arrow[r, "\alpha"'] & L
\end{tikzcd}$$

DIALECTICA PETRI NETS

truncated subtraction

$(\mathbb{N}, +, 0, \ominus, \geq)$ is a lineale

pre-conditions

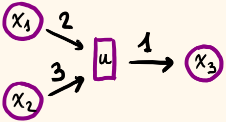
post-conditions

transitions

places

CATEGORY $\mathrm{Net}_{\mathbb{N}}$

- objects are $({}^\bullet\alpha, \alpha^\bullet)$ with ${}^\bullet\alpha, \alpha^\bullet : U \times X \to \mathbb{N}$ in $\mathcal{S}et$

$\rightsquigarrow$ 

${}^\bullet\alpha(u, x_1) = 2 \qquad {}^\bullet\alpha(u, x_2) = 1$

$\alpha^\bullet(u, x_3) = 1$

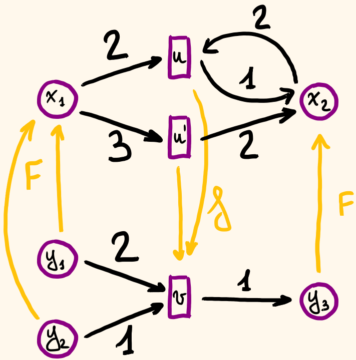
- morphisms are $(f, F) : ({}^\bullet\alpha, \alpha^\bullet) \to ({}^\bullet\beta, \beta^\bullet)$

with $\begin{cases} (f, F) : (U, X, {}^\bullet\alpha) \longrightarrow (V, Y, {}^\bullet\beta) \\ (f, F) : (U, X, \alpha^\bullet) \longrightarrow (V, Y, \beta^\bullet) \end{cases}$ in $\mathrm{Dial}_{\mathbb{N}}$

MORPHISMS

$$\begin{tikzcd} U\times Y \arrow[r, "g\times\mathbb{1}"] \arrow[d, "1\times F"'] \arrow[dr, phantom, "\Nearrow"] & V\times Y \arrow[d, "\beta"] \\ U\times X \arrow[r, "\alpha"'] & L \end{tikzcd}$$

$$\begin{tikzcd} ({}^\bullet\alpha, \alpha^\bullet) \arrow[d, "{(g,F)}"'] \\ ({}^\bullet\beta, \beta^\bullet) \end{tikzcd}$$



OUTLINE

- PART 0 : the Dialectica construction
- [PART 1 : linear logic structure]
- PART 2 : changing the arcs

LINEAR LOGIC STRUCTURE ON NETS

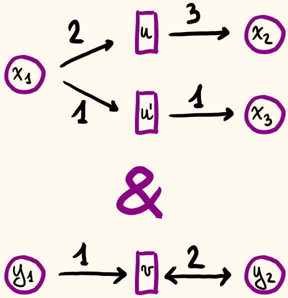
- cartesian product $\&$
- coproduct $\oplus$
- monoidal product $\otimes$
- internal hom $[-,-]$

CARTESIAN PRODUCT

$$({}^\bullet\alpha, \alpha^\bullet) \;\&\; ({}^\bullet\beta, \beta^\bullet) := ({}^\bullet\alpha \,\&\, {}^\bullet\beta, \; \alpha^\bullet \,\&\, \beta^\bullet)$$

where $\alpha \,\&\, \beta : U \times V \times (X+Y) \longrightarrow \mathbb{N}$

$$(u,v,x) \longmapsto \alpha(u,x)$$
$$(u,v,y) \longmapsto \beta(v,y)$$



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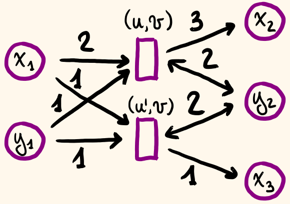
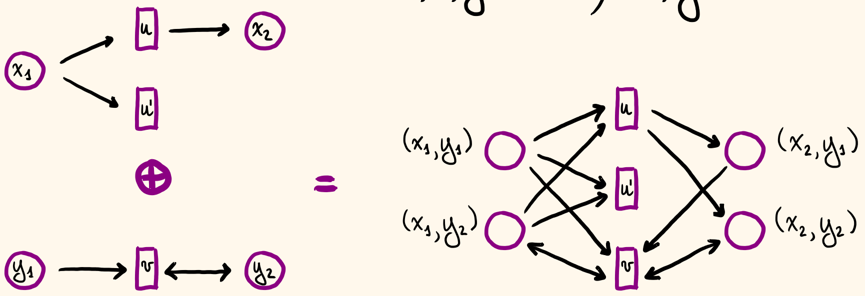


COPRODUCT

$$({}^{\bullet}\alpha, \alpha^{\bullet}) \oplus ({}^{\bullet}\beta, \beta^{\bullet}) := ({}^{\bullet}\alpha \oplus {}^{\bullet}\beta, \alpha^{\bullet} \oplus \beta^{\bullet})$$

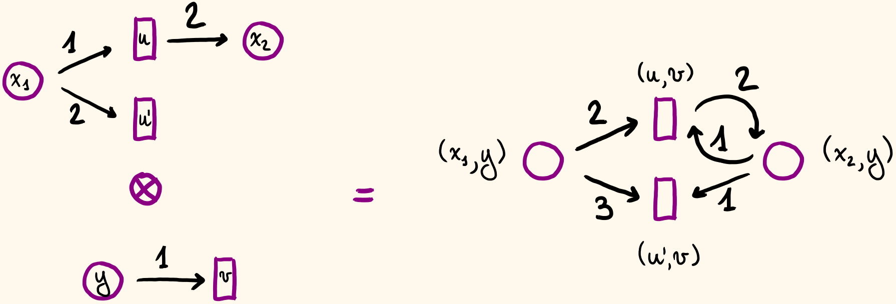
where
$$\begin{aligned} \alpha \oplus \beta : (U+V) \times X \times Y &\longrightarrow \mathbb{N} \\ (u,x,y) &\longmapsto \alpha(u,x) \\ (v,x,y) &\longmapsto \beta(v,y) \end{aligned}$$



MONOIDAL PRODUCT

$$({}^\bullet\alpha, \alpha^\bullet) \otimes ({}^\bullet\beta, \beta^\bullet) := ({}^\bullet\alpha \otimes {}^\bullet\beta, \alpha^\bullet \otimes \beta^\bullet)$$

where
$$\begin{aligned} \alpha \otimes \beta : U \times V \times X^V \times Y^U &\longrightarrow \mathbb{N} \\ (u, v, f, g) &\longmapsto \alpha(u, f(v)) + \beta(v, g(u)) \end{aligned}$$

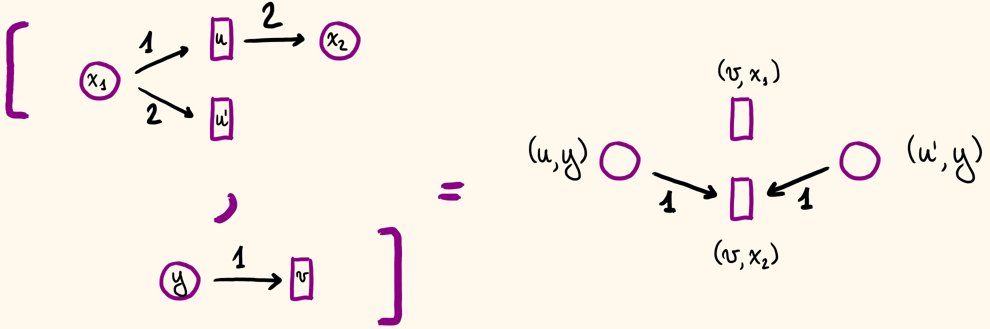


INTERNAL HOM

$$[({}^\bullet\alpha,\alpha^\bullet),({}^\bullet\beta,\beta^\bullet)] := ([{}^\bullet\alpha,{}^\bullet\beta],[\alpha^\bullet,\beta^\bullet])$$

where $[\alpha,\beta] : V^U \times X^Y \times U \times Y \longrightarrow \mathbb{N}$

$$(g,F,u,y) \longmapsto \beta(g(u),y) \ominus \alpha(u,F(y))$$



OUTLINE

- PART 0 : the Dialectica construction
- PART 1 : linear logic structure
- [PART 2 : changing the arcs]

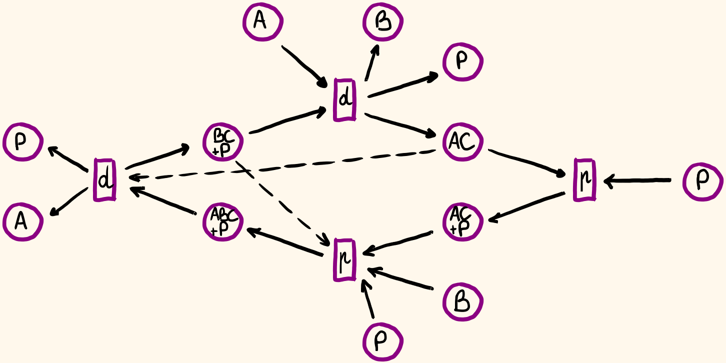
CHANGING THE LINEALE

- $L = 3$ ⇝ uncertain arcs
- $L = [0, 1]$ ⇝ arcs with probabilities
- $L = \mathbb{R}^+$ ⇝ arcs with rates
- $L = \mathbb{Z}$ ⇝ inhibitor arcs
- $L = L_1 \times L_2$ ⇝ product of lineales

PETRI NETS WITH UNCERTAINTY

$(3, \min, 1, \multimap, \leq)$ is a lineale

$a \multimap b := \max\{x : \min\{x, a\} \leq b\}$

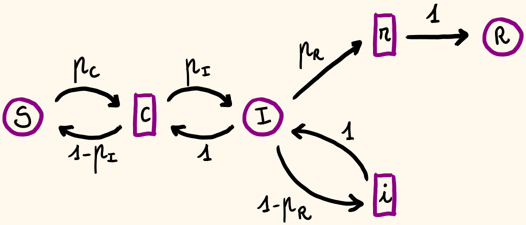


[3] Axmann, Legewie, Herzel, A minimal circadian clock model, 2007

PROBABILISTIC PETRI NETS

$([0,1], \cdot, 1, \multimap, \leq)$ is a lineale

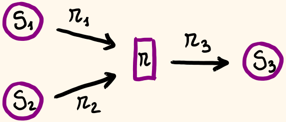
$$a \multimap b := \begin{cases} b/a & a \geq b \wedge a \neq 0 \\ 1 & a < b \vee a = 0 \end{cases}$$



PETRI NETS WITH RATES

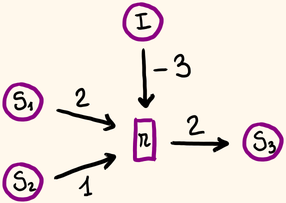
$(\mathbb{R}^+, +, 0, \ominus, \geq)$ is a lineale

truncated subtraction



PETRI NETS WITH INHIBITORS

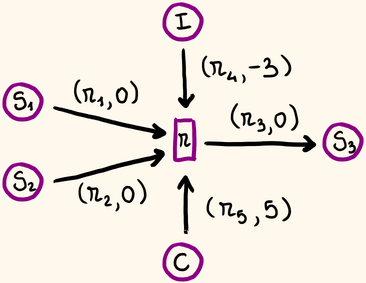
$(\mathbb{Z}, +, 0, -, \leq)$ is a lineale



PRODUCT OF LINEALES

$(L_1, *_1, e_1, \multimap_1, \leq_1)$ and $(L_2, *_2, e_2, \multimap_2, \leq_2)$ lineales

$\Rightarrow (L_1 \times L_2, *, (e_1, e_2), \multimap, \leq)$ is a lineale



$$\rightsquigarrow \begin{cases} L_1 = \mathbb{R}^+ \\ L_2 = \mathbb{Z} \end{cases}$$

CONCLUSIONS & FUTURE WORK

- linear logic can be useful to combine nets

FUTURE WORK

- behaviour of nets ?
- implementations ?

THANKS FOR LISTENING !