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DIALECTICA PETRI NETS

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MOTIVATION & INTRODUCTION

- combine Petri nets using linear logic connectives

- Dialectica construction [1]
  - → models of linear logic
  - → category of Petri nets [2]

[1] V. de Paiva, The Dialectica categories, PhD thesis 1991
[2] C. Brown and D. Gurr, A categorical linear framework for Petri nets, 1995

OUTLINE

- **PART 0** : the Dialectica construction
- **PART 1** : structure on nets
- **PART 2** : changing the arcs

LINEALES

$(L, *, e, \multimap, \leq)$ is a monoidal closed poset :

- $(L, *, e)$ monoid
- $\leq$ partial order on L compatible with $*$

  $a \leq a', b \leq b' \Rightarrow a * b \leq a' * b'$
- $\multimap$ internal hom w.r.t. $*$

  $a \leq a', b \leq b' \Rightarrow a' \multimap b \leq a \multimap b'$

  $b * c \leq a \Leftrightarrow b \leq c \multimap a$ $\quad (- * c \dashv c \multimap -)$

⤳ lift the structure of L to L-valued relations

LINEALES - EXAMPLES

- $(2=\{0,1\}, \wedge, 1, \to, \leq)$
- $(\mathbb{N}, +, 0, \ominus, \geq)$

  $\ominus$ ↳ truncated subtraction
- $(3=\{-1,0,1\}, \min, 1, \multimap, \leq)$

  $$\multimap \; \begin{cases} a \leq b \Rightarrow a \multimap b = a \\ b \geq a \Rightarrow a \multimap b = b \end{cases}$$
- $([0,1], \cdot, 1, \multimap, \leq)$

  $$\multimap \; \begin{cases} a \geq b \wedge a \neq 0 \Rightarrow a \multimap b = b/a \\ a < b \vee a = 0 \Rightarrow a \multimap b = 1 \end{cases}$$
- $(\mathbb{R}^{\geq 0}, +, 0, \ominus, \geq)$

  $\ominus$ ↳ truncated subtraction
- $(\mathbb{Z}, +, 0, -, \leq)$

DIALECTICA CONSTRUCTION

DIALECTICA CATEGORY $\mathfrak{Dial}_L$

- objects : $\alpha : U \times X \to L$ in $\mathcal{S}et$ ⇝ 'L-valued relations'
- morphisms : $(f, F) : \alpha \to \beta$ are

$$\begin{tikzcd} U \arrow[r, "f"] & V \\ X & Y \arrow[l, "F"'] \end{tikzcd} \quad \text{such that} \quad \begin{tikzcd} U \times Y \arrow[r, "f \times Y"] \arrow[d, "U \times F"'] \arrow[dr, phantom, "\leq"] & V \times Y \arrow[d, "\beta"] \\ U \times X \arrow[r, "\alpha"'] & L \end{tikzcd}$$

- composition : $(f, F) ; (g, G) := (f ; g, G ; F)$
- identities : $id_\alpha := (id_U, id_X)$

EXAMPLE

$L = (\mathbb{N}, +, 0, \ominus, \geq)$

$U = \{u_1, u_2\}$ $X = \{x\}$

$V = \{v\}$ $Y = \{y_1, y_2\}$

$\alpha(u_1, x) = 2$
$\alpha(u_2, x) = 1$
$\beta(v, y_1) = 0$
$\beta(v, y_2) = 1$

$(f, F) : \alpha \to \beta$

$f(u_i) := v$

$F(y_i) := x$

$$\begin{tikzcd}
(u_1, y_1) \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & (v, y_1) \arrow[ddd, mapsto] \\
& U \times Y \arrow[r, "f \times Y"] \arrow[d, "U \times F"'] & V \times Y \arrow[d, "\beta"] & \\
& U \times X \arrow[r, "\alpha"'] \arrow[ur, phantom, "\Nearrow"] & L & \\
(u_1, x) \arrow[rrr, mapsto] & & & 2 \geq 0
\end{tikzcd}$$

COMPARISON WITH THE ORIGINAL DIALECTICA

$\mathcal{C}$ category with

- finite limits
- finite coproducts that are disjoint and stable under pullback

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, hook, "i_Y"] \\
X \arrow[r, hook, "i_X"] & X+Y
\end{tikzcd}
\qquad
\begin{tikzcd}
A_j^* \arrow[r, "f_j^*"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "f"] \\
A_j \arrow[r, hook, "i_j"] & A
\end{tikzcd}$$

$A := A_1 + \cdots + A_n$

s.t. $f_1^* + \cdots + f_n^*$ is iso

- locally cartesian closed ($= \mathcal{C}/X$ cartesian closed)

⇝ lift the structure of $\mathcal{C}$ to relations in $\mathcal{C}$

DIALECTICA CATEGORY G$\mathcal{C}$ [1]

- objects : monos $\alpha : A \hookrightarrow U \times X$ in $\mathcal{C}$ $\rightsquigarrow$ 'relations in $\mathcal{C}$'
- morphisms : $(f, F) : \alpha \to \beta$ are

$$U \xrightarrow{f} V$$
$$X \xleftarrow{F} Y$$

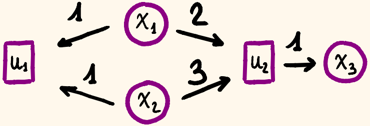
such that

$$\begin{tikzcd}
 & \bar{B} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, hook, "\beta"] \\
\bar{A} \arrow[ur, dashed, "\exists k"] \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U \times Y \arrow[r, "f \times Y"'] \arrow[d, "U \times F"] & V \times Y \\
A \arrow[r, hook, "\alpha"'] & U \times X &
\end{tikzcd}$$

- composition : $(f, F) ; (g, G) := (f ; g, G ; F)$
- identities : $id_\alpha := (id_U, id_X)$

PETRI NETS

- X set of places
- U set of transitions
- ${}^\bullet\alpha : U \times X \longrightarrow \mathbb{N}$ preconditions
- $\alpha^\bullet : U \times X \longrightarrow \mathbb{N}$ postconditions



$X = \{x_1, x_2, x_3\}$

$U = \{u_1, u_2\}$

	${}^\bullet\alpha$	$\alpha^\bullet$
(u_1, x_1)	1	0
(u_2, x_1)	2	0
(u_1, x_2)	1	0
(u_2, x_2)	3	0
(u_1, x_3)	0	0
(u_2, x_3)	0	1

DIALECTICA PETRI NETS

CATEGORY $\mathbf{Net}_L$

- objects : $(\overset{\text{pre-conditions}}{{}^\bullet\alpha}, \overset{\text{post-conditions}}{\alpha^\bullet})$ with ${}^\bullet\alpha, \alpha^\bullet : \overset{\text{transitions}}{U} \times \overset{\text{places}}{X} \longrightarrow L$ in $\mathbf{Set}$

  are pairs of objects in $\mathbf{Dial}_L$

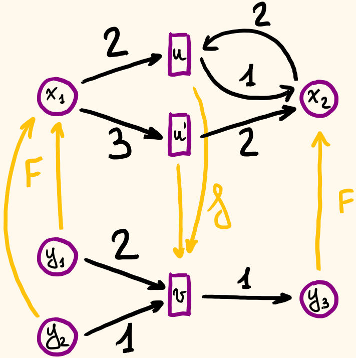
  $\rightsquigarrow L = \mathbb{N} \Rightarrow$ Petri nets

- morphisms : $(f, F) : ({}^\bullet\alpha, \alpha^\bullet) \longrightarrow ({}^\bullet\beta, \beta^\bullet)$

  with $\begin{cases} (f, F) : {}^\bullet\alpha \longrightarrow {}^\bullet\beta \\ (f, F) : \alpha^\bullet \longrightarrow \beta^\bullet \end{cases}$ in $\mathbf{Dial}_L$

MORPHISMS

$$\begin{tikzcd}
({}^\bullet\alpha, \alpha^\bullet) \arrow[d, "{(g,F)}"'] \\
({}^\bullet\beta, \beta^\bullet)
\end{tikzcd}$$



$$\begin{tikzcd}
U \times Y \arrow[r, "g \times 1"] \arrow[d, "1 \times F"'] \arrow[dr, phantom, "\Nearrow"] & V \times Y \arrow[d, "\beta"] \\
U \times X \arrow[r, "\alpha"'] & L
\end{tikzcd}$$

PETRI NETS IN $G\mathcal{C}$ [2]

CATEGORY GNet

- objects: $(\overset{\text{pre-conditions}}{{}^\circ\alpha}, \overset{\text{post-conditions}}{\alpha^\circ})$ with ${}^\circ\alpha, \alpha^\circ : A \hookrightarrow \overset{\text{transitions}}{U} \times \overset{\text{places}}{X}$ in $\mathcal{C}$

  are pairs of objects in $G\mathcal{C}$

  ⇝ $\mathcal{C} = \mathbf{Set}$ ⇒ *elementary Petri nets*

- morphisms: $(f, F) : ({}^\circ\alpha, \alpha^\circ) \to ({}^\circ\beta, \beta^\circ)$

  with $$\begin{cases} (f,F) : {}^\circ\alpha \to {}^\circ\beta \\ (f,F) : \alpha^\circ \to \beta^\circ \end{cases} \quad \text{in } G\mathcal{C}$$

VARIATIONS OVER MORPHISMS

- morphisms $(f, F) : ({}^{\bullet}\alpha, \alpha^{\bullet}) \to ({}^{\bullet}\beta, \beta^{\bullet})$ could be defined by

$$\begin{cases} (f, F) : {}^{\bullet}\alpha \to {}^{\bullet}\beta \\ (f, F) : \beta^{\bullet} \to \alpha^{\bullet} \end{cases} \quad \text{in } \mathcal{D}ial_L$$

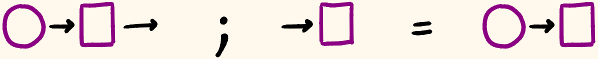
OUTLINE

- PART 0 : the Dialectica construction
- PART 1 : structure on nets
- PART 2 : changing the arcs

COMBINE VS COMPOSE PETRI NETS

- composing Petri nets along places [3]

- composing Petri nets along transitions [4]



- combining Petri nets: nets are objects, not morphisms

[3] J. Baez and J. Master, Open Petri nets, 2020
[4] J. Ratke, P. Sobociński and O. Stephens, Decomposing Petri nets, 2013

STRUCTURE ON NETS

• cartesian product	$\&$
• coproduct	$\oplus$
• monoidal product	$\otimes$
• internal hom	$[-,-]$
• (par)	$\parr$

CARTESIAN PRODUCT

$$\alpha : U \times X \longrightarrow L$$
$$\beta : V \times Y \longrightarrow L$$

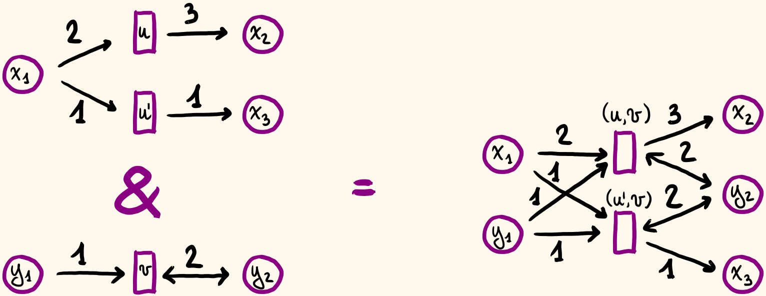
$$\alpha \,\&\, \beta \; : U \times V \times (X + Y) \longrightarrow L$$

$$\alpha \,\&\, \beta := \begin{pmatrix} \alpha \times \varepsilon_V \\ \beta \times \varepsilon_U \end{pmatrix}$$

$$(u, v, x) \longmapsto \alpha(u, x)$$
$$(u, v, y) \longmapsto \beta(v, y)$$

$$({}^{\bullet}\alpha, \alpha^{\bullet}) \;\&\; ({}^{\bullet}\beta, \beta^{\bullet}) := ({}^{\bullet}\alpha \,\&\, {}^{\bullet}\beta, \alpha^{\bullet} \,\&\, \beta^{\bullet})$$

CARTESIAN PRODUCT - EXAMPLE



⇝ synchronisation between the nets

CARTESIAN PRODUCT IN $G\mathcal{C}$

$$\alpha : A \hookrightarrow U \times X$$
$$\beta : B \hookrightarrow V \times Y$$

$$\alpha \,\&\, \beta : (A \times V) + (B \times U) \hookrightarrow U \times V \times (X + Y)$$

$$\alpha \,\&\, \beta := (\alpha \times id_V) + (\beta \times id_U)$$

$\mathcal{C} = \mathcal{S}et$:

$$(u,v)\,(\alpha \,\&\, \beta)\; x \iff u \,\alpha\, x$$
$$(u,v)\,(\alpha \,\&\, \beta)\; y \iff v \,\beta\, y$$

COPRODUCT

$$\alpha : U \times X \longrightarrow L$$
$$\beta : V \times Y \longrightarrow L$$

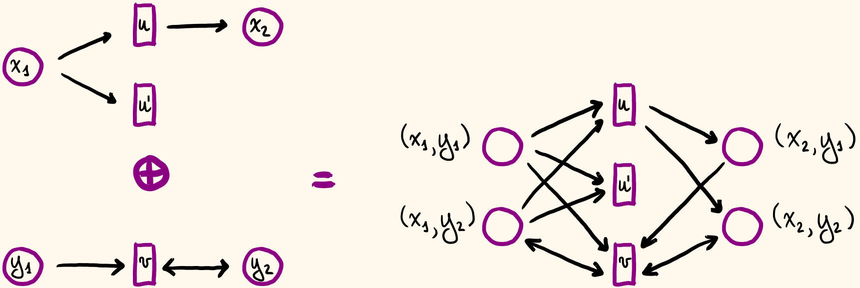
$$\alpha \oplus \beta \quad : (U+V) \times X \times Y \longrightarrow L$$

$$\alpha \oplus \beta := \begin{pmatrix} \alpha \times \varepsilon_Y \\ \beta \times \varepsilon_X \end{pmatrix}$$

$$\begin{aligned} (u,x,y) &\longmapsto \alpha(u,x) \\ (v,x,y) &\longmapsto \beta(v,y) \end{aligned}$$

$$({}^{\bullet}\alpha, \alpha^{\bullet}) \oplus ({}^{\bullet}\beta, \beta^{\bullet}) := ({}^{\bullet}\alpha \oplus {}^{\bullet}\beta, \alpha^{\bullet} \oplus \beta^{\bullet})$$

COPRODUCT – EXAMPLE



⇝ resource sharing between the nets

COPRODUCT IN $\mathcal{GC}$

$\alpha : A \hookrightarrow U \times X$
$\beta : B \hookrightarrow V \times Y$

$\alpha \oplus \beta : (A \times Y) + (B \times X) \hookrightarrow (U + V) \times X \times Y$

$\alpha \oplus \beta := (\alpha \times id_Y) + (\beta \times id_X)$

$\mathcal{C} = \mathcal{S}et :$

$u \, (\alpha \oplus \beta) \, (x, y) \iff u \, \alpha \, x$

$v \, (\alpha \oplus \beta) \, (x, y) \iff v \, \beta \, y$

MONOIDAL PRODUCT

$$\alpha : U \times X \longrightarrow L$$
$$\beta : V \times Y \longrightarrow L$$

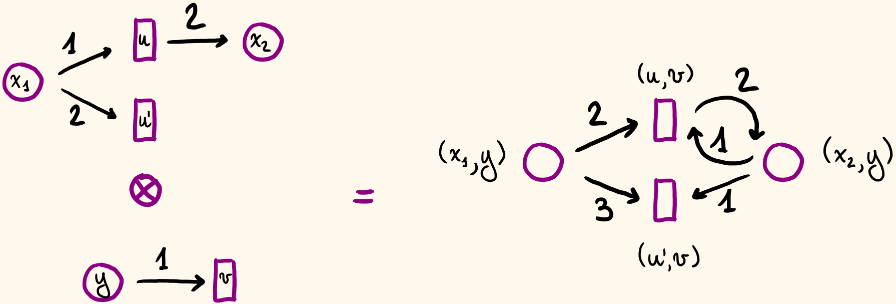
$$\alpha \otimes \beta \; : U \times V \times X^V \times Y^U \longrightarrow L$$

$$\alpha \otimes \beta := U \times V \times X^V \times Y^U \xrightarrow{\text{copy}} U \times V \times U \times V \times X^V \times Y^U$$
$$\xrightarrow{\text{evaluate}} U \times X \times V \times Y \xrightarrow{\alpha \times \beta} L \times L \xrightarrow{*} L$$

$$(u, v, f, g) \longmapsto \alpha(u, f(v)) * \beta(v, g(u))$$

$$({}^{\bullet}\alpha, \alpha^{\bullet}) \otimes ({}^{\bullet}\beta, \beta^{\bullet}) := ({}^{\bullet}\alpha \otimes {}^{\bullet}\beta, \alpha^{\bullet} \otimes \beta^{\bullet})$$

MONOIDAL PRODUCT - EXAMPLE



MONOIDAL PRODUCT IN $G\mathcal{C}$

$$\alpha : A \hookrightarrow U \times X$$

$$\beta : B \hookrightarrow V \times Y$$

$$\alpha \otimes \beta : A \otimes B \longrightarrow U \times V \times X^V \times Y^U$$

$$\begin{tikzcd}
A \otimes B \arrow[r, hook] \arrow[d, hook] \arrow[dr, "\alpha \otimes \beta"] \arrow[dr, phantom, "\lrcorner", very near start] & \bar{B} \arrow[rr] \arrow[d, hook] \arrow[drr, phantom, "\lrcorner", very near start] & & B \arrow[d, hook, "\beta"] \\
\bar{A} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U \times V \times X^V \times Y^U \arrow[rr, "V \times ev \times \varepsilon"] \arrow[d, "U \times ev \times \varepsilon"] & & V \times Y \\
A \arrow[r, hook, "\alpha"'] & U \times X & &
\end{tikzcd}$$

$\mathcal{C} = \mathcal{S}et$:

$$(u, v)\,(\alpha \otimes \beta)\,(f, g) \iff u \;\alpha\; f(v) \;\wedge\; v \;\beta\; g(u)$$

INTERNAL HOM

$$\alpha : U \times X \longrightarrow L$$
$$\beta : V \times Y \longrightarrow L$$

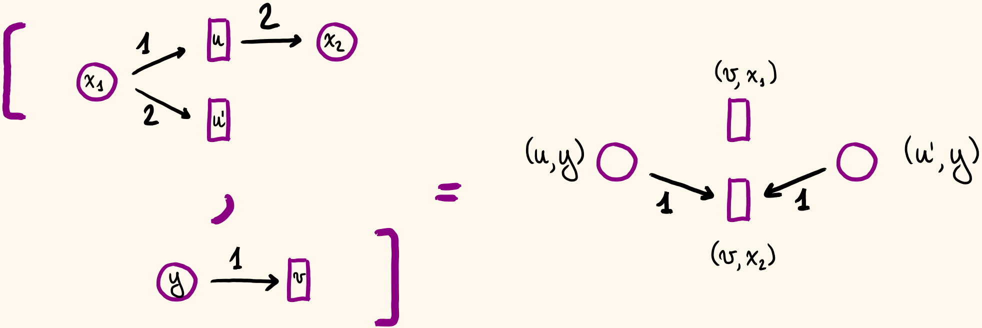
$$[\alpha, \beta] : V^U \times X^Y \times U \times Y \longrightarrow N$$

$$[\alpha, \beta] := V^U \times X^Y \times U \times Y \xrightarrow{\text{copy}} V^U \times X^Y \times U \times Y \times U \times Y$$
$$\xrightarrow{\text{evaluation}} U \times X \times V \times Y \xrightarrow{\alpha \times \beta} L \times L \xrightarrow{\multimap} L$$

$$(g, F, u, y) \longmapsto \alpha(u, F(y)) \multimap \beta(g(u), y)$$

$$[({}^\bullet\alpha, \alpha^\bullet), ({}^\bullet\beta, \beta^\bullet)] := ([{}^\bullet\alpha, {}^\bullet\beta], [\alpha^\bullet, \beta^\bullet])$$

INTERNAL HOM - EXAMPLE



INTERNAL HOM IN $G\mathcal{C}$

$\alpha : A \hookrightarrow U \times X$ $\beta : B \hookrightarrow V \times Y$

$[\alpha,\beta] : [A,B] \hookrightarrow V^U \times X^Y \times U \times Y$ is the greatest subobject s.t. $[\alpha,\beta] \wedge \bar{\alpha} \leq \bar{\beta}$

$$\begin{tikzcd}
P \arrow[r, hook] \arrow[d, hook] \arrow[dr, "{[\alpha,\beta]\wedge\bar{\alpha}}"] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[rr, hook, bend left] & {[A,B]} \arrow[d, hook, "{[\alpha,\beta]}"] & \bar{B} \arrow[dl, "\bar{\beta}"'] \arrow[dr] \arrow[d, phantom, "\vee"] & \\
\bar{A} \arrow[r, hook, "\bar{\alpha}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & V^U \times X^Y \times U \times Y \arrow[d, "\varepsilon \times ev \times U"] \arrow[dr, "ev \times \varepsilon \times Y"] & & B \arrow[dl, "\beta"] \\
A \arrow[r, hook, "\alpha"'] & U \times X & V \times Y &
\end{tikzcd}$$

$\mathcal{C} = \mathcal{S}et$:

$$(f,F)\,[\alpha,\beta]\,(u,y) \iff u\,\alpha\,F(y) \to f(u)\,\beta\,y$$

PAR IN $G\mathcal{C}$

$\alpha : A \hookrightarrow U \times X \qquad \beta : B \hookrightarrow V \times Y$

$\alpha \parr \beta : \bar{A} + \bar{B} \hookrightarrow U^Y \times V^X \times X \times Y$

$$\begin{tikzcd}
\bar{A}+\bar{B} \arrow[drr, phantom, ""] \arrow[dr, "\alpha \parr \beta"] & \bar{B} \arrow[l, hook', "i_{\bar{B}}"'] \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, hook, "\beta"] \\
\bar{A} \arrow[u, hook, "i_{\bar{A}}"] \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U^Y \times V^X \times X \times Y \arrow[r, "\varepsilon \times ev \times Y"'] \arrow[d, "ev \times \varepsilon \times X"] & V \times Y \\
A \arrow[r, hook, "\alpha"'] & U \times X &
\end{tikzcd}$$

$\mathcal{C} = \mathcal{S}et :$

$$(f,g)\,(\alpha \parr \beta)\,(x,y) \iff f(y)\,\alpha\,x \;\vee\; g(x)\,\beta\,y$$

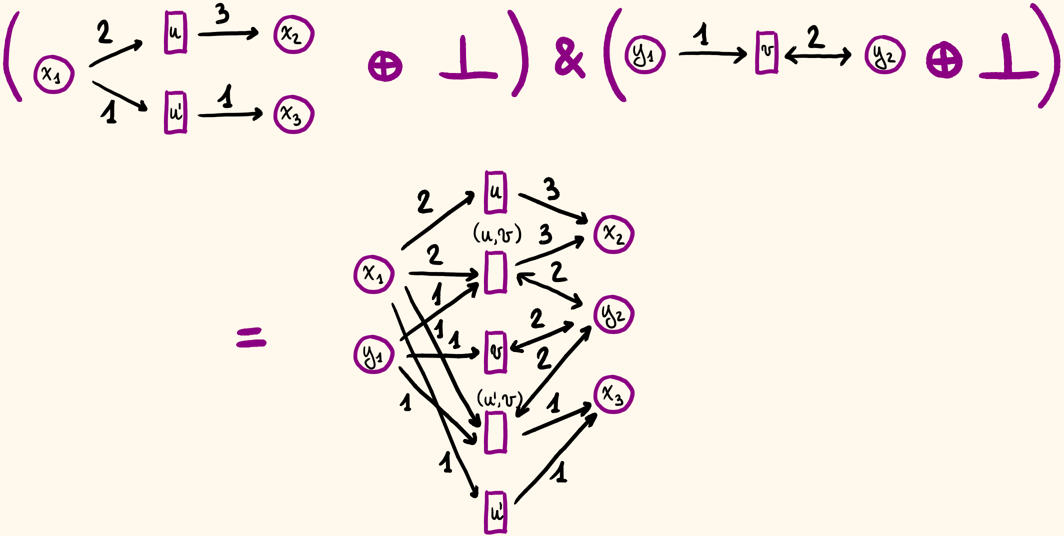
UNITS AND NEGATION

- $1 = \square$ terminal object
- $0 = \bigcirc$ initial object
- $I = \bigcirc \circlearrowright \square$ monoidal unit for $\otimes$
- $\perp = \bigcirc \quad \square$ monoidal unit for ⅋

$\rightsquigarrow \neg \alpha := [\alpha, \perp]$

$$\neg\ \bigcirc \rightarrow \square \leftarrow \bigcirc \quad = \quad \bigcirc \leftarrow \square \rightarrow \bigcirc$$

ASYNCHRONOUS EVENTS



OUTLINE

- PART 0 : the Dialectica construction
- PART 1 : structure on nets

[• PART 2 : changing the arcs]

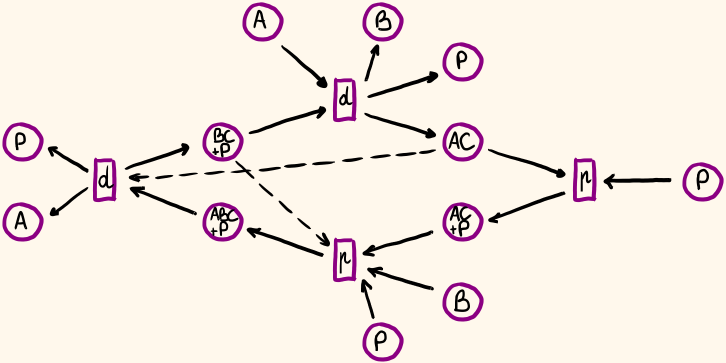
CHANGING THE LINEALE

- $L = 3$ ⇝ uncertain arcs
- $L = [0,1]$ ⇝ arcs with probabilities
- $L = \mathbb{R}^{\geq 0}$ ⇝ arcs with rates
- $L = \mathbb{Z}$ ⇝ inhibitor arcs
- $L = L_1 \times L_2$ ⇝ product of lineales

PETRI NETS WITH UNCERTAINTY

$(3, \min, 1, \multimap, \leq)$ is a lineale

$\hookrightarrow a \multimap b := \max\{x : \min\{x, a\} \leq b\}$

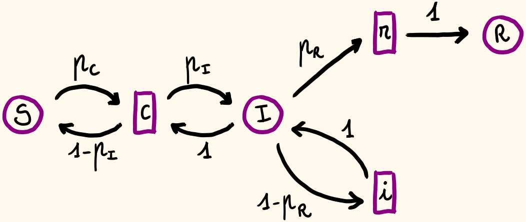


[5] Axmann, Legewie, Herzel, A minimal circadian clock model, 2007

PROBABILISTIC PETRI NETS

$([0,1], \cdot, 1, \multimap, \leq)$ is a lineale

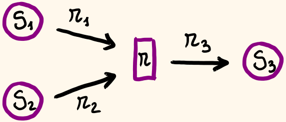
$$a \multimap b := \begin{cases} b/a & a \geq b \wedge a \neq 0 \\ 1 & a < b \vee a = 0 \end{cases}$$



PETRI NETS WITH RATES

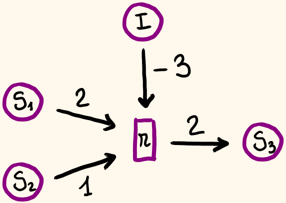
$(\mathbb{R}^{\geq 0}, +, 0, \ominus, \geq)$ is a lineale

truncated subtraction



PETRI NETS WITH INHIBITORS

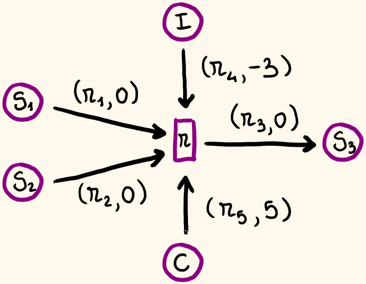
$(\mathbb{Z}, +, 0, -, \leq)$ is a lineale



PRODUCT OF LINEALES

$(L_1, *_1, e_1, \multimap_1, \leq_1)$ and $(L_2, *_2, e_2, \multimap_2, \leq_2)$ lineales

$\Rightarrow (L_1 \times L_2, *, (e_1, e_2), \multimap, \leq)$ is a lineale



$$\rightsquigarrow \begin{cases} L_1 = \mathbb{R}^{\geq 0} \\ L_2 = \mathbb{Z} \end{cases}$$

CONCLUSIONS & FUTURE WORK

- linear logic can be useful to combine nets

FUTURE WORK

- behaviour of nets ?
- implementations ?

THANKS FOR LISTENING !

DIALECTICA AS MODEL FOR LINEAR LOGIC

THEOREM [de Paiva, 1991]

$\mathcal{C}$ has finite limits, has finite disjoint coproducts that are stable under pullback, and is locally cartesian closed

$\Rightarrow$ $G\mathcal{C}$ is a sound model of intuitionistic linear logic

$\forall \alpha \in G\mathcal{C} \quad \alpha \simeq \neg\neg \alpha$

$\Rightarrow$ $G\mathcal{C}$ is $*$-autonomous and a sound model of classical linear logic