PLAN FOR TODAY

17/4/2024

PART 1

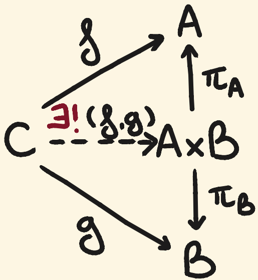
- recap cartesian categories
- (symmetric) monoidal categories
- string diagrams (& do-notation)
- monoids, comonoids & modules
- Fox's theorem

PART 2

- (symmetric) monoidal functors
- coherence & strictification (overview)
- (symmetric) monoidal natural transformations

RECAP: CARTESIAN CATEGORIES

CARTESIAN PRODUCT



⇝ $A \times B$ is the resource A in parallel with the resource B

CARTESIAN CATEGORY

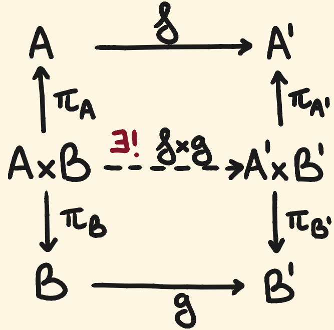
$\mathcal{C}$ category with

- a terminal object 1
- for all objects A and B, their cartesian product $A \times B$

RECAP: PROPERTIES OF CARTESIAN PRODUCTS

CARTESIAN PRODUCT FUNCTOR

$$(- \times =) : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$$
$$(A, B) \longmapsto A \times B$$
$$(f, g) \longmapsto f \times g$$



⇝ $f \times g$ is the process that executes f and g in parallel

ASSOCIATIVITY & UNITALITY

$$\alpha_{A,B,C} : A \times (B \times C) \xrightarrow{\cong} (A \times B) \times C$$

$\lambda_A : A \xrightarrow{\cong} 1 \times A$ and $\rho_A : A \xrightarrow{\cong} A \times 1$

MONOIDAL CATEGORIES : INTRO

Cartesian categories provide an algebra of
sequential $f: A \to B,\ g: B \to C \Rightarrow f;g: A \to C$
and parallel $f: A \to B,\ f': A' \to B' \Rightarrow f \times f': A \times A' \to B \times B'$
composition.

Monoidal categories generalise this aspect of cartesian categories.

⇝ object represent resources
arrows represent processes/computations

MONOIDAL CATEGORIES

STRICT MONOIDAL CATEGORY

$(\mathcal{C}, \otimes, I)$

- $\mathcal{C}$ category
- $(- \otimes =) : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ functor (tensor/monoidal product)
- $I \in \mathrm{obj}\,\mathcal{C}$ (monoidal unit)

such that

- $A \otimes (B \otimes C) = (A \otimes B) \otimes C$ (associativity)
- $A \otimes I = A$ (unitality)
  $I \otimes A = A$

⇝ the objects of $\mathcal{C}$ form a monoid
the arrows of $\mathcal{C}$ form a monoid

MONOIDAL CATEGORIES

MONOIDAL CATEGORY

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$

- $\mathcal{C}$ category
- $(- \otimes =) : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ functor (tensor/monoidal product)
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- $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$ (associativity)
- $A \otimes I \cong A$ (unitality)
  $I \otimes A \cong A$

→ natural isomorphisms + coherence conditions

MONOIDAL CATEGORIES

MONOIDAL CATEGORY

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$

- $\mathcal{C}$ category
- $(- \otimes =) : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ functor (tensor/monoidal product)
- $I \in \mathrm{obj}\,\mathcal{C}$ (monoidal unit)
- natural isomorphisms

$\alpha : - \otimes (= \otimes \equiv) \Rightarrow (- \otimes =) \otimes \equiv$ (associator)

$\lambda : id_{\mathcal{C}} \Rightarrow I \otimes id_{\mathcal{C}}$ (left and right unitors)

$\rho : id_{\mathcal{C}} \Rightarrow id_{\mathcal{C}} \otimes I$

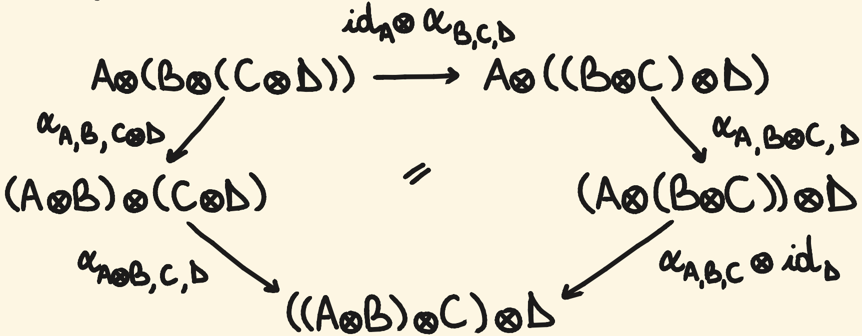
such that

(⬠) (△)

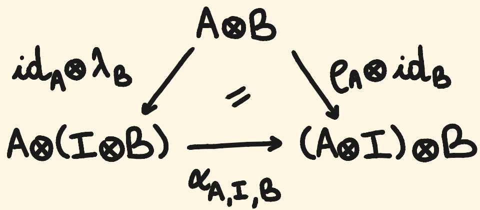
MONOIDAL CATEGORIES

α, λ, ρ such that

(⬠) PENTAGON EQUATION



(Δ) TRIANGLE EQUATION



MONOIDAL CATEGORIES : EXAMPLES

- cartesian categories $(\mathcal{C}, \times, 1)$ es. $(\mathcal{S}et, \times, 1)$
  $\rightsquigarrow f \times g$ is f and g
- cocartesian categories $(\mathcal{C}, +, 0)$ es. $(\mathcal{S}et, +, \emptyset)$
  $\rightsquigarrow f + g$ is f or g
- $(\mathcal{C}at(\mathcal{C}, \mathcal{C}), \circ, id_{\mathcal{C}})$
- $(Ab, \otimes, \mathbb{Z})$ of abelian groups with their tensor product

MONOIDAL CATEGORIES: MORE EXAMPLES

- $(\mathrm{Mon}, \otimes, I)$ of monoids (details later)

- $(\mathrm{Mod}, \otimes, I)$ of modules (details later)

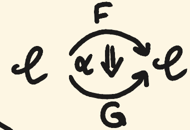
- Linear maps between vector or Hilbert spaces
  $(\mathrm{Vect}, \otimes, I)$ and $(\mathrm{Hilb}, \otimes, I)$
  ⇝ **PROJECT IDEA:** Heunen & Kornell (2022)
  Axioms for the category of Hilbert spaces

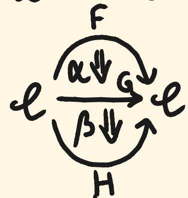
- Many more examples next time
  ↳ $(\mathrm{Par}, \times, 1)$, $(\mathrm{Rel}, \times, 1)$, ...

ENDOFUNCTORS

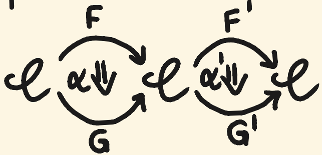
$\mathcal{C}at(\mathcal{C},\mathcal{C})$:

objects are functors $F : \mathcal{C} \to \mathcal{C}$

morphisms are natural transformations 

composition is vertical composition 

monoidal product is horizontal composition



monoidal unit is the identity functor

ENDOFUNCTORS

PROPOSITION

$(\mathcal{C}at(\mathcal{C},\mathcal{C}), \circ, id_{\mathcal{C}})$ is a strict monoidal category.

PROOF

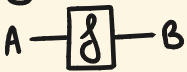
Exercise. $\square$

STRING DIAGRAMS

Convenient syntax for monoidal categories.

Arrows are boxes with possibly many or no inputs/outputs.

$f : A \to B$

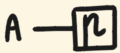


$h : A \otimes B \to C \otimes D \otimes E$

$s : I \to A$

state

$r : A \to I$



predicate / effect / costate

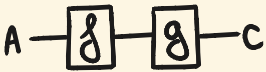
$id_A : A \to A$

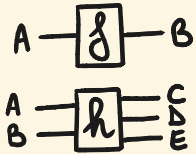
Joyal & Street (1991) The geometry of tensor calculus I

STRING DIAGRAMS

Sequential and parallel composition

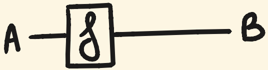
$f;g : A \to C$ $\qquad$ $f \otimes h : A \otimes A \otimes B \to B \otimes C \otimes D \otimes E$





Structural equations become trivial

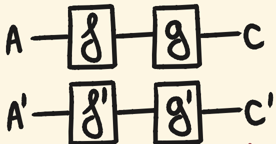
$f; id_B =$



$=$

$= f$

$(f \otimes f');(g \otimes g') =$



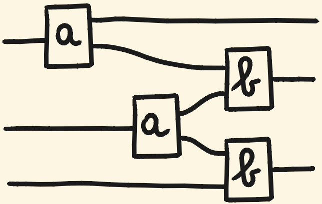
$= (f;g) \otimes (f';g')$

← interchange law

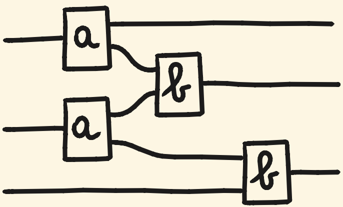
STRING DIAGRAMS ARE NICE

$(a \otimes id \otimes id);(id \otimes id \otimes a \otimes id);(id \otimes b \otimes b)$

=



=



$= (a \otimes a \otimes id);(id \otimes b \otimes id \otimes id);(id \otimes id \otimes b)$

STRING DIAGRAMS ARE FORMAL

THEOREM

a well-typed equation between arrows in a monoidal category
is a consequence of the axioms iff
it holds in string diagrams up to planar isotopy.

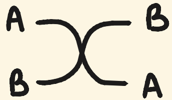
⇝ more on this later

SYMMETRIES

In some monoidal categories, resources can be permuted.

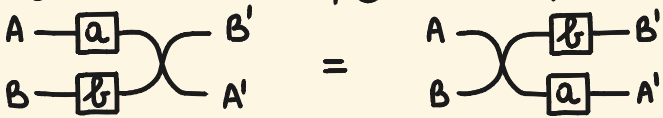
For all objects A and B, there is a symmetry

$$\sigma_{A,B} : A \otimes B \longrightarrow B \otimes A$$

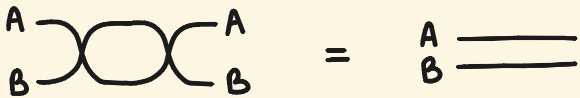


SYMMETRIES

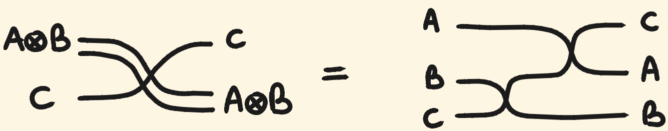
Symmetries satisfy some equations



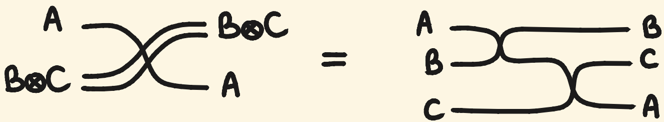
(naturality)



(invertibility)



(⬡ equations)

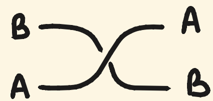


INTERLUDE : BRAIDINGS

In some monoidal categories, the way of permuting resources matters.

$\sigma_{A,B} : A \otimes B \longrightarrow B \otimes A$

$\sigma^{-1}_{A,B} : B \otimes A \longrightarrow A \otimes B$



SYMMETRIC MONOIDAL CATEGORIES

SYMMETRIC MONOIDAL CATEGORY

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \sigma)$

- $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ monoidal category
- $\sigma : (- \otimes =) \Rightarrow (= \otimes -)$ natural isomorphism

such that

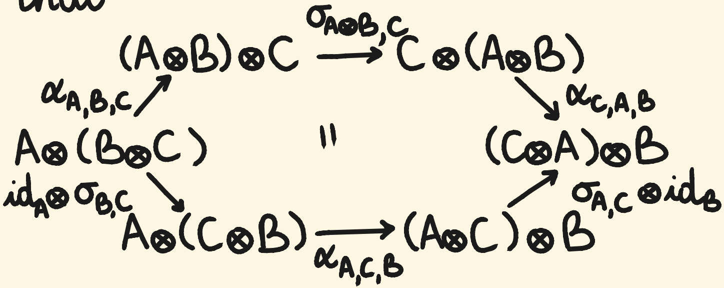
(⬡1) (⬡2)

and $\sigma_{B,A} = \sigma^{-1}_{A,B}$. ⇝ drop this for braided

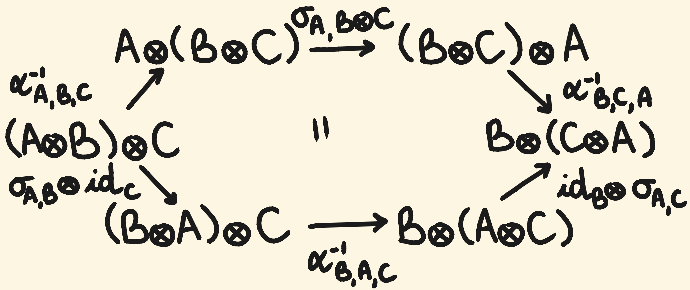
SYMMETRIC MONOIDAL CATEGORIES

σ such that

(1)

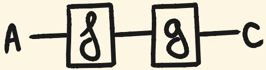


(2)

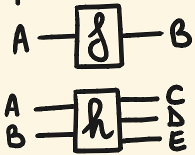


SMCs : SUMMARY

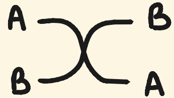
sequential

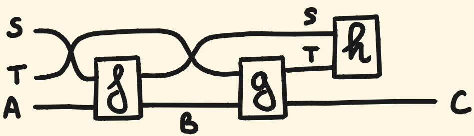


parallel



symmetries





$$(\sigma_{S,T} \otimes id_A);(id_T \otimes f);(\sigma_{T,S} \otimes id_B);(id_S \otimes g);(h \otimes id_C)$$

MONOIDS

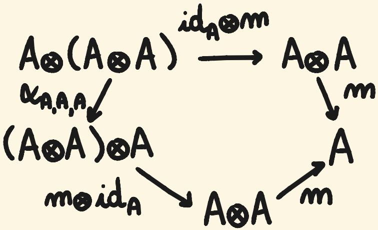
$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ monoidal category.

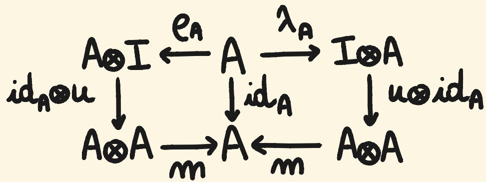
MONOID

(A, m, u)

- $A \in \text{ob}\,\mathcal{C}$ (carrier)
- $m : A \otimes A \to A$ (multiplication)
- $u : I \to A$ (unit)

that are associative and unital





MONOIDS

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ monoidal category.

MONOID

(A, m, u)

- $A \in \mathrm{obj}\,\mathcal{C}$ (carrier)
- 

or 

(multiplication)
- 

or 

(unit)

that are associative and unital

MONOIDS : EXAMPLES

- Monoids in $(\mathbf{Set}, \times, 1)$ are set-theoretic monoids.

<u>ex</u> $(\mathbb{N}, +, 0)$ is a monoid
$(A^*, ::, ())$ is a monoid

- In $(\mathbf{Set}, +, \varnothing)$ there is a canonical monoid on every A

$$[id, id] : A + A \longrightarrow A \qquad (a, i) \longmapsto a \qquad\qquad j : \varnothing \longrightarrow A$$

- Monoids in $\mathbf{Cat}(\mathcal{C}, \mathcal{C})$ are monads (T, μ, η).

- Monoids in the monoidal category $(\mathbf{Ab}, \otimes, \mathbb{Z})$ of abelian groups are rings.

COMONOIDS

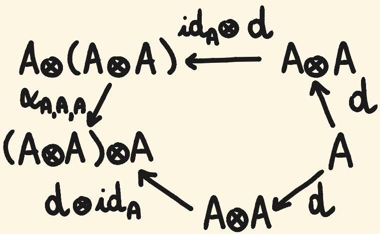
$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ monoidal category.

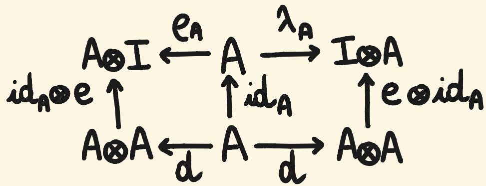
COMONOID (= monoid in $\mathcal{C}^{op}$)

(A, d, e)

- $A \in \mathrm{obj}\,\mathcal{C}$ (carrier)
- $d : A \longrightarrow A \otimes A$ (comultiplication)
- $e : A \longrightarrow I$ (counit)

that are coassociative and counital



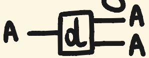


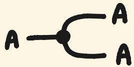
COMONOIDS

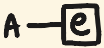
$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ monoidal category.

COMONOID (= monoid in $\mathcal{C}^{op}$)

(A, d, e)

- $A \in \mathrm{obj}\,\mathcal{C}$ (carrier)
- 

or 

(comultiplication)
- 

or 

(counit)

that are coassociative and counital

COMONOIDS : EXAMPLES

- In $(\mathbf{Set}, \times, 1)$ there is a canonical comonoid on every A

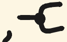
$$\langle id, id \rangle : A \longrightarrow A \times A \qquad\qquad ! : A \longrightarrow 1$$
$$a \longmapsto (a,a) \qquad\qquad a \longmapsto *$$

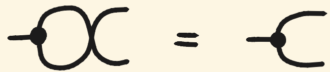
- Comonoids in $\mathbf{Cat}(\mathcal{C},\mathcal{C})$ are comonads (T, δ, ε).

COMMUTATIVITY

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \sigma)$ symmetric monoidal category.

A monoid (A, , ) is commutative if

A comonoid (A, , ) is cocommutative if



ex $(\mathbb{N}, +, 0)$

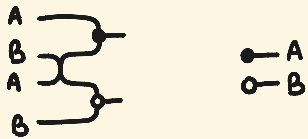
$(A, \langle id, id \rangle, !)$

NON-ex monads

comonads

COMPOSING (CO)MONOIDS

In a symmetric monoidal category, monoids compose.
For two monoids (A, ⊃•−, •−) and (B, ⊃∘−, ∘−),
there is a monoid on $A \otimes B$



⚠ when the monoidal category is not symmetric,
we need distributive laws.

⇝ **PROJECT IDEA**: distributive laws of monads & liftings

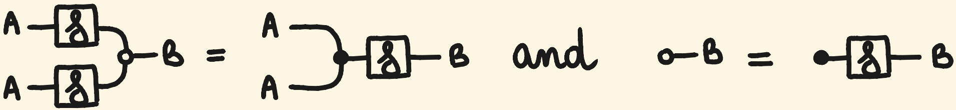
MORPHISMS OF (CO)MONOIDS

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ monoidal category.

$(A$, ⊃•—, •—$)$ and $(B$, ⊃∘—, ∘—$)$ monoids.

MONOID MORPHISM

$f : (A$, ⊃•—, •—$) \longrightarrow (B$, ⊃∘—, ∘—$)$ is an arrow $f : A \to B$ such that



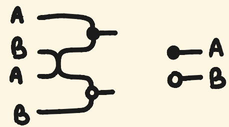
CATEGORY OF MONOIDS

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \sigma)$ symmetric monoidal category.

$\mathrm{Mon}(\mathcal{C})$:

objects are monoids

arrows are morphisms of monoids

monoidal product is composition of monoids 

monoidal unit is the monoid on I 

$= \rho_I^{-1} = \lambda_I^{-1}$

CATEGORY OF MONOIDS

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \sigma)$ symmetric monoidal category.

PROPOSITION

$\mathrm{Mon}(\mathcal{C})$ is a symmetric monoidal category.

PROOF

Exercise. $\square$

MODULES

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ monoidal category.

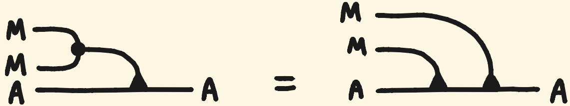
$(M, ⊃—, •—)$ monoid.

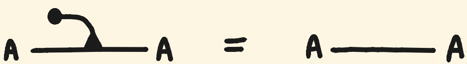
MODULE OVER A MONOID

A module (A, a) over a monoid $(M, ⊃—, •—)$ is

- $A \in \mathrm{obj}\,\mathcal{C}$ (carrier)
- $a : M \otimes A \longrightarrow A$ (action)

such that





MODULE MORPHISMS

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ monoidal category.

(M, ⊸-, ○-) monoid.

(A, ⊥) and (B, ⊥) modules over (M, ⊸-, ○-).

MODULE MORPHISM

$f : (A, ⊥) \longrightarrow (B, ⊥)$ is

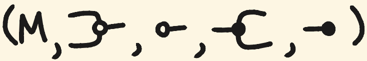
$f : A \to B$ such that

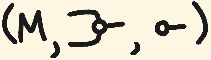
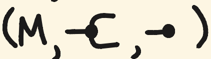
BIALGEBRAS

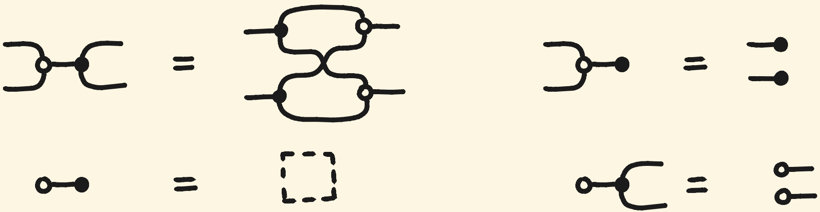
Modules over bialgebras can be tensored.

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \sigma)$ symmetric monoidal category.

BIALGEBRA (or BIMONOID)

 is a bialgebra if

-  monoid
-  comonoid
- bialgebra axioms (= the monoid is a comonoid morphism
  = the comonoid is a monoid morphism)



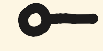
BIALGEBRAS : EXAMPLES

- $(\mathbb{N}, +, 0, \text{copy}, \text{discard})$
- any monoid in Set, with copy and discard
- any monoid in a cartesian category, with copy and discard
- the category $(\mathrm{Ab}, \oplus, 0)$ of abelian groups with their direct product is cartesian (and cocartesian)

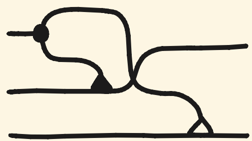
PRODUCT OF MODULES

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \sigma)$ symmetric monoidal category.

(M, , , , ) bialgebra.

(A, ) and (B, ) modules over (M, , ).

There is a module on $A \otimes B$ whose action is



NOTE: this is not the tensor product of modules, which requires coequalisers in $\mathcal{C}$.

CATEGORY OF MODULES

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \sigma)$ symmetric monoidal category.

$(M, \succ\!\!\circ\!-, \circ\!-, -\!\!\bullet\!\!\prec, -\!\bullet)$ bialgebra.

PROPOSITION

Modules over $(M, \succ\!\!\circ\!-, \circ\!-)$ form a symmetric monoidal category $(M\text{-Mod}, \otimes, I)$.

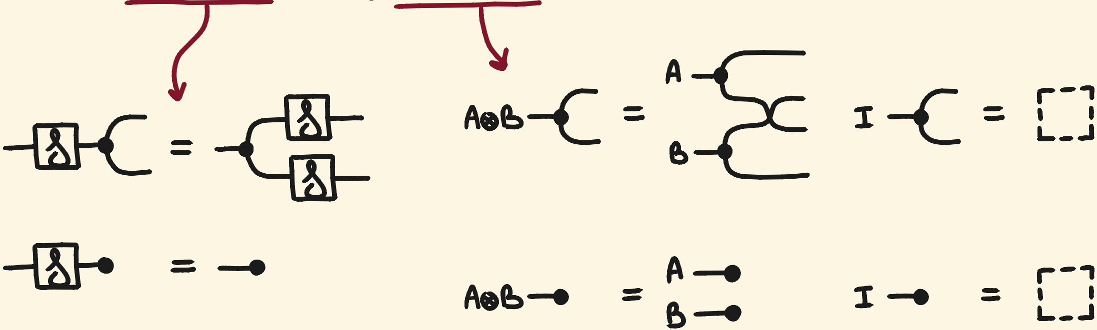
PROOF

Exercise. □

FOX'S THEOREM

THEOREM

A symmetric monoidal category is cartesian iff every object has a natural and coherent comonoid structure.



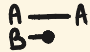
Fox (1976) Coalgebras and cartesian categories

FOX'S THEOREM

PROOF SKETCH

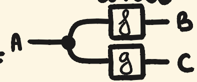
(⇐) The maps to the terminal object are $!_A :=$ 

.

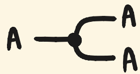
The projection maps are $\pi_A :=$ 

and $\pi_B :=$ 

.

The pairing maps are $\langle f, g \rangle :=$ 

.

(⇒) The copy maps are 

$:= \langle id_A, id_A \rangle$.

The discard maps are 

$:= !_A$.

Naturality follows from the universal properties. □

COPY-DISCARD CATEGORIES

CD-CATEGORY

A copy-discard category is a symmetric monoidal category where every object has a coherent cocommutative comonoid structure.

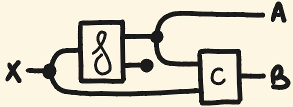
<u>ex</u> (Par, ×, 1), (Rel, ×, 1)
+ more next time

⇝ PROJECT IDEAS: cartesian restriction categories
(partial) Markov categories
cartesian bicategories of relations

Corradini & Gadducci (1999)

INTERLUDE: DO-NOTATION

Morphisms in copy-discard categories can be specified by do-notation.




```
cond(x) = do
  f(x) → (a, b)
  c(a, x) → b'
  return (a, b')
```

⇝ PROJECT IDEA: premonoidal categories & do-notation
premonoidal vs monoidal categories

PART 1

- recap cartesian categories
- (symmetric) monoidal categories
- string diagrams (& do-notation)
- monoids, comonoids & modules
- Fox's theorem

PART 2

- (symmetric) monoidal functors
- coherence & strictification (overview)
- (symmetric) monoidal natural transformations

MONOIDAL FUNCTORS

$(\mathcal{C}, \otimes, I)$ and $(\mathcal{D}, \boxtimes, J)$ strict monoidal categories

STRICT MONOIDAL FUNCTOR

$F : (\mathcal{C}, \otimes, I) \longrightarrow (\mathcal{D}, \boxtimes, J)$ is
a functor $F : \mathcal{C} \to \mathcal{D}$ such that

- $J = F(I)$
- $F(A) \boxtimes F(B) = F(A \otimes B)$

⇝ strict monoidal functors are functors that strictly preserve the monoidal structure

MONOIDAL FUNCTORS

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{D}, \boxtimes, J, \alpha', \lambda', \rho')$ monoidal categories

STRONG MONOIDAL FUNCTOR

$(F, e, m) : (\mathcal{C}, \otimes, I, \alpha, \lambda, \rho) \longrightarrow (\mathcal{D}, \boxtimes, J, \alpha', \lambda', \rho')$ is
a functor $F : \mathcal{C} \to \mathcal{D}$ such that

- $J \cong F(I)$
- $F(A) \boxtimes F(B) \cong F(A \otimes B)$

→ natural isomorphisms
+ coherence conditions

⇝ strong monoidal functors are functors that preserve the monoidal structure up to isomorphism

MONOIDAL FUNCTORS

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{D}, \boxtimes, J, \alpha', \lambda', \rho')$ monoidal categories

STRONG MONOIDAL FUNCTOR

$(F, e, m) : (\mathcal{C}, \otimes, I, \alpha, \lambda, \rho) \longrightarrow (\mathcal{D}, \boxtimes, J, \alpha', \lambda', \rho')$ is

- a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$
- natural isomorphisms $e : J \longrightarrow F(I)$
  and $m : F(-) \boxtimes F(=) \longrightarrow F(- \otimes =)$

that are associative and unital.

MONOIDAL FUNCTORS

e, m such that

- associativity

$$
\begin{array}{ccc}
F(A) \boxtimes (F(B) \boxtimes F(C)) & \xrightarrow{\alpha'_{FA,FB,FC}} & (F(A) \boxtimes F(B)) \boxtimes F(C) \\
\downarrow {\scriptstyle id_{FA} \boxtimes m_{B,C}} & & \downarrow {\scriptstyle m_{A,B} \boxtimes id_{FC}} \\
F(A) \boxtimes (F(B \otimes C)) & = & (F(A \otimes B)) \boxtimes F(C) \\
\downarrow {\scriptstyle m_{A,B \otimes C}} & & \downarrow {\scriptstyle m_{A \otimes B, C}} \\
F(A \otimes (B \otimes C)) & \xrightarrow[F\alpha_{A,B,C}]{} & F((A \otimes B) \otimes C)
\end{array}
$$

- unitality

$$
\begin{array}{ccc}
J \boxtimes F(A) & \xrightarrow{e \boxtimes id_{FA}} & F(I) \boxtimes F(A) \\
\uparrow {\scriptstyle \lambda'_{FA}} & = & \downarrow {\scriptstyle m_{I,A}} \\
F(A) & \xrightarrow[F(\lambda_A)]{} & F(I \otimes A)
\end{array}
\qquad
\begin{array}{ccc}
F(A) \boxtimes J & \xrightarrow{id_{FA} \boxtimes e} & F(A) \boxtimes F(I) \\
\uparrow {\scriptstyle \rho'_{FA}} & = & \downarrow {\scriptstyle m_{I,A}} \\
F(A) & \xrightarrow[F(\rho_A)]{} & F(A \otimes I)
\end{array}
$$

MONOIDAL FUNCTORS

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{D}, \boxtimes, J, \alpha', \lambda', \rho')$ monoidal categories

LAX MONOIDAL FUNCTOR

$(F, e, m) : (\mathcal{C}, \otimes, I, \alpha, \lambda, \rho) \longrightarrow (\mathcal{D}, \boxtimes, J, \alpha', \lambda', \rho')$ is

- a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$
- natural transformations $e : J \longrightarrow F(I)$
  and $m : F(-) \boxtimes F(=) \longrightarrow F(- \otimes =)$

that are associative and unital.

$\rightsquigarrow$ lax monoidal functors are functors that laxly preserve the monoidal structure

CATEGORIES OF MONOIDAL CATEGORIES

- StrMonCat of strict monoidal categories
  and strict monoidal functors

- MonCat of monoidal categories
  and strong monoidal functors

MONOIDAL CATEGORIES : STRICTIFICATION

MONOIDAL EQUIVALENCE

A monoidal equivalence is a monoidal functor whose underlying functor is an equivalence.

THEOREM (STRICTIFICATION OF MONOIDAL CATEGORIES)

Every monoidal category $\mathcal{C}$ is monoidally equivalent to a strict one $S(\mathcal{C})$.

$\rightsquigarrow$ we can forget about α, λ, ρ

MONOIDAL CATEGORIES : COHERENCE

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ monoidal category

$\mathcal{C}_S \subseteq \mathcal{C}$ smallest monoidal subcategory of $\mathcal{C}$ containing, for all objects A, B and C, $\alpha_{A,B,C}$, λ_A and ρ_A

THEOREM (COHERENCE FOR MONOIDAL CATEGORIES)

For any two objects X and Y in $\mathcal{C}_S$, the set $\mathcal{C}_S(X,Y)$ either

- is empty, or
- contains exactly one element

$\rightsquigarrow$ any well-typed equation between arrows in $\mathcal{C}_S$ holds

COHERENCE & STRICTIFICATION IN PRACTICE

Take a monoidal category $\mathcal{C}$.

Suppose we have $f, g : X \to Y$ in $\mathcal{C}$ and we want to show $f = g$.

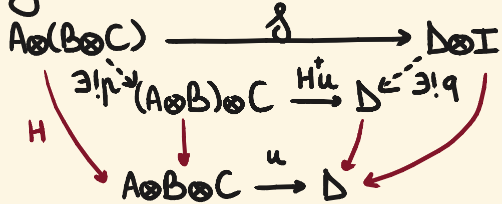
Consider the equivalence $H : \mathcal{C} \to S(\mathcal{C})$.

Consider $u := H(f)$ and $v := H(g)$ in $S(\mathcal{C})$.

If we show $u = v$, we obtain $f = g$ because

$\exists!$ iso $p : X \to H^{+}H(X)$ and $q : Y \to H^{+}H(Y)$ by coherence

$\Rightarrow f = p ; H^{+}u ; q^{-1} = p ; H^{+}v ; q^{-1} = g$



STRING DIAGRAMS ARE FREE

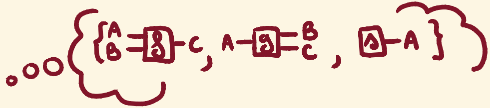
$$\mathbf{MonSig} \underset{U}{\overset{\text{STRING}}{\rightleftarrows}} \mathbf{StrMonCat} \underset{\text{STRICT}}{\overset{U}{\rightleftarrows}} \mathbf{MonCat}$$

with $\perp$ between MonSig and StrMonCat, and $\top$ between StrMonCat and MonCat.

String diagrams are constructed over a monoidal signature

MONOIDAL SIGNATURE

$$\Sigma = (E \overset{s,t}{\rightrightarrows} V^*)$$



a morphism $h : \Sigma \to \Sigma'$ is $(h_E : E \to E', h_V : V \to V')$ such that

$$\begin{array}{ccc} E & \xrightarrow{h_E} & E' \\ s \downarrow & & \downarrow s' \\ V^* & \xrightarrow[h_V^*]{} & V'^* \end{array} \qquad \begin{array}{ccc} E & \xrightarrow{h_E} & E' \\ t \downarrow & & \downarrow t' \\ V^* & \xrightarrow[h_V^*]{} & V'^* \end{array}$$

RECAP: STRING DIAGRAMS ARE FORMAL

String diagrams over $U(S(\mathcal{C}))$ are sound and complete for $S(\mathcal{C})$

Equalities in $S(\mathcal{C})$ show equalities in $\mathcal{C}$

$\Rightarrow$ Equalities of string diagrams show equalities in $\mathcal{C}$

SYMMETRIC MONOIDAL FUNCTORS

$(\mathcal{C}, \otimes, I)$ and $(\mathcal{D}, \boxtimes, J)$ symmetric monoidal categories

SYMMETRIC MONOIDAL FUNCTOR

A monoidal functor $(F, e, m) : (\mathcal{C}, \otimes, I) \longrightarrow (\mathcal{D}, \boxtimes, J)$ is symmetric if

$$\begin{array}{ccc} F(A) \boxtimes F(B) & \xrightarrow{m_{A,B}} & F(A \otimes B) \\ {\scriptstyle \sigma'_{FA,FB}} \downarrow & = & \downarrow {\scriptstyle F\sigma_{A,B}} \\ F(B) \boxtimes F(A) & \xrightarrow[m_{B,A}]{} & F(B \otimes A) \end{array}$$

COHERENCE & STRICTIFICATION

As for monoidal categories, but we need to take care of permutations: $\sigma_{A,A} \neq id_A \otimes id_A$ $\left({}^{A}_{A}\!\times \neq {}^{A}_{A}\!= \right)$.

→ all formal well-typed equations between morphisms constructed by $\alpha, \lambda, \rho, \sigma$ and with the same underlying permutation hold

STRICTIFICATION (SMCs)

Every symmetric monoidal category is symmetric monoidally equivalent to a symmetric strict one.

MONOIDAL NATURAL TRANSFORMATIONS

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{D}, \boxtimes, J, \alpha', \lambda', \rho')$ monoidal categories

$(F, m, e), (G, m', e') : (\mathcal{C}, \otimes, I) \to (\mathcal{D}, \boxtimes, J)$ monoidal functors

MONOIDAL NATURAL TRANSFORMATION

$\beta : (F, m, e) \Rightarrow (G, m', e')$ is

a natural transformation $\beta : F \Rightarrow G$

such that

$$\begin{array}{ccc} F(A) \boxtimes F(B) & \xrightarrow{\beta_A \boxtimes \beta_B} & G(A) \boxtimes G(B) \\ {\scriptstyle m_{A,B}} \downarrow & = & \downarrow {\scriptstyle m'_{A,B}} \\ F(A \otimes B) & \xrightarrow[\beta_{A \otimes B}]{} & G(A \otimes B) \end{array}$$

$$\begin{array}{ccc} & J & \\ {\scriptstyle e} \swarrow & = & \searrow {\scriptstyle e'} \\ F(I) & \xrightarrow[\beta_I]{} & G(I) \end{array}$$

When F, G are symmetric, so is β.