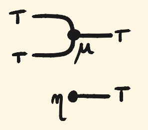
MONADS

MONAD (♥)

A monad on a category $\mathcal{C}$ is a triple (T, μ, η) of
- a functor $T : \mathcal{C} \to \mathcal{C}$
- a natural transformation $\mu : T;T \to T$ (multiplication)
- a natural transformation $\eta : \mathbb{1}_{\mathcal{C}} \to T$ (unit)



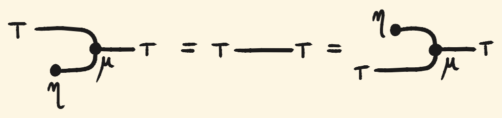
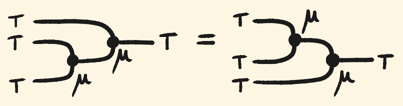
such that, for all $A \in \mathcal{C}_0$

$$\eta_{TA} ; \mu_A = \mathbb{1}_{TA} = T\eta_A ; \mu_A \qquad \mu_{TA} ; \mu_A = T\mu_A ; \mu_A$$

$$\begin{tikzcd} TA \arrow[r, "\eta_{TA}"] \arrow[dr, "\mathbb{1}_{TA}"'] & TTA \arrow[d, "\mu_A"] \arrow[dl, phantom, "="] \arrow[dr, phantom, "="] & TA \arrow[l, "T\eta_A"'] \arrow[dl, "\mathbb{1}_{TA}"] \\ & TA & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} TTTA \arrow[r, "\mu_{TA}"] \arrow[d, "T\mu_A"'] \arrow[dr, phantom, "="] & TTA \arrow[d, "\mu_A"] \\ TTA \arrow[r, "\mu_A"] & TA \end{tikzcd}$$

(unitality) (associativity)

i.e. $(\mathbb{1}_T \otimes \eta) ; \mu = \mathbb{1}_T = (\eta \otimes \mathbb{1}_T) ; \mu$ and $(\mathbb{1}_T \otimes \mu) ; \mu = (\mu \otimes \mathbb{1}_T) ; \mu$

⇝ monads on $\mathcal{C}$ are monoids in the (monoidal) category $[\mathcal{C}, \mathcal{C}]$ of endofunctors on $\mathcal{C}$ and natural transformations between them.

MONAD IN KLEISLI FORM (useful for programming)

A monad on a category $\mathcal{C}$ is a triple $(T, (-)^T, \eta)$ of
- a function $T : \mathcal{C}_0 \to \mathcal{C}_0$
- an operation $(-)^T : \mathcal{C}(A, TB) \to \mathcal{C}(TA, TB)$ (Kleisli extension)
- a family of morphisms $\eta_A : A \to TA$ in $\mathcal{C}$, for every $A \in \mathcal{C}_0$ (unit)

such that

1. $(\eta_A)^T = \mathbb{1}_{TA}$
2. $\eta_A ; f^T = f$
3. $f^T ; g^T = (f ; g^T)^T$

PROPOSITION

The two definitions of monad are equivalent.

PROOF

(⇒) Suppose we have a monad (T, μ, η).
We want to define a Kleisli extension for it.
For $f : A \to TB$, define $f^T := T(f) ; \mu_B : TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB$.
We need to show 1, 2 & 3.

1. $\eta_A^T := T(\eta_A) ; \mu_A = \mathbb{1}_{TA}$ by unitality
2. $\eta_A ; f^T := \eta_A ; T(f) ; \mu_B$
   $= f ; \eta_{TB} ; \mu_B$ by naturality of η
   $= f ; \mathbb{1}_{TB}$ by unitality
   $= f$ by identity laws
3. $f^T ; g^T := T(f) ; \mu_B ; T(g) ; \mu_C$
   $= T(f) ; TT(g) ; \mu_{TC} ; \mu_C$ by naturality of μ
   $= T(f) ; TT(g) ; T\mu_C ; \mu_C$ by associativity
   $= T(f ; Tg ; \mu_C) ; \mu_C$ by functoriality of T
   $= T(f ; g^T) ; \mu_C$
   $= (f ; g^T)^T$

⇒ $(T, (-)^T, \eta)$ is a monad in Kleisli form.

(⇐) Suppose we have a monad $(T, (-)^T, \eta)$ in Kleisli form.
We can define the action of T on morphisms as $T(f) := (f ; \eta_B)^T$, for $f : A \to B$.
This defines a functor:

$T(f) ; T(g) := (f ; \eta_B)^T ; (g ; \eta_C)^T$ by definition of T
$= (f ; \eta_B ; (g ; \eta_C)^T)^T$ by 3.
$= (f ; g ; \eta_C)^T$ by 2.
$=: T(f ; g)$ by definition of T

$T(\mathbb{1}_A) := (\mathbb{1}_A ; \eta_A)^T$ by definition of T
$= \eta_A^T$ by identity laws
$= \mathbb{1}_{TA}$ by 1.

We need to show that the family of morphisms $\{\eta_A\}_{A \in \mathcal{C}_0}$ is natural.
For $f : A \to B$,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\eta_A"'] \arrow[dr, phantom, "="] & B \arrow[d, "\eta_B"] \\ TA \arrow[r, "Tf"'] & TB \end{tikzcd}$$

because $\eta_A ; Tf := \eta_A ; (f ; \eta_B)^T$ by definition of T
$= f ; \eta_B$ by 2.

We can define a multiplication $\mu_A := (\mathbb{1}_{TA})^T : TT(A) \to T(A)$.
We need to show naturality, associativity and unitality.

- Naturality.
  For $f : A \to B$,

$$\begin{tikzcd} TTA \arrow[r, "TTf"] \arrow[d, "\mu_A"'] \arrow[dr, phantom, "="] & TTB \arrow[d, "\mu_B"] \\ TA \arrow[r, "Tf"'] & TB \end{tikzcd}$$

because

$TTf ; \mu_B := ((f ; \eta_B)^T ; \eta_{TB})^T ; \mathbb{1}_{TB}^T$ by definitions of μ and T
$= ((f ; \eta_B)^T ; \eta_{TB} ; \mathbb{1}_{TB}^T)^T$ by 3.
$= ((f ; \eta_B)^T ; \mathbb{1}_{TB})^T$ by 2.
$= (\mathbb{1}_{TA} ; (f ; \eta_B)^T)^T$ by identity laws
$= \mathbb{1}_{TA}^T ; (f ; \eta_B)^T$ by 3.
$=: \mu_A ; Tf$ by definitions of μ and T

- Associativity.

$\mu_{TA} ; \mu_A$
$:= (\mathbb{1}_{TTA})^T ; (\mathbb{1}_{TA})^T$ by def of T and μ
$= (\mathbb{1}_{TTA} ; \mathbb{1}_{TA}^T)^T$ by 3.
$= \mathbb{1}_{TA}^{TT}$ by identities
$=: \mu_A^T$ by def of μ

$T\mu_A ; \mu_A$
$:= (\mathbb{1}_{TA}^T ; \eta_{TA})^T ; \mathbb{1}_{TA}^T$ by def of T and μ
$= (\mathbb{1}_{TA}^T ; \eta_{TA} ; \mathbb{1}_{TA}^T)^T$ by 3.
$= (\mathbb{1}_{TA}^T ; \mathbb{1}_{TA})^T$ by 2.
$= \mathbb{1}_{TA}^{TT}$ by identities
$=: \mu_A^T$ by def of μ

- Unitality.

$\eta_{TA} ; \mu_A$
$:= \eta_{TA} ; \mathbb{1}_{TA}^T$ by def of μ
$= \mathbb{1}_{TA}$ by 2.

$T\eta_A ; \mu_A$
$:= (\eta_A ; \eta_{TA})^T ; \mathbb{1}_{TA}^T$ by def of T and μ
$= (\eta_A ; \eta_{TA} ; \mathbb{1}_{TA}^T)^T$ by 3.
$= (\eta_A ; \mathbb{1}_{TA})^T$ by 2.
$= \eta_A^T$ by identities
$= \mathbb{1}_{TA}$ by 1.

⇒ (T, μ, η) is a monad. □

ADJUNCTIONS & MONADS

SLOGAN: *Every adjunction gives a monad.*

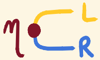
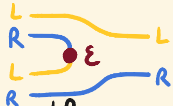
THEOREM

For an adjunction $\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$ ($\perp$)*, the endofunctor* $L;R : \mathcal{C} \to \mathcal{C}$ *has a monad structure.*

PROOF

We have the unit  *and the counit*  *of the adjunction.*

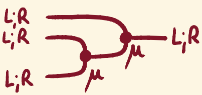
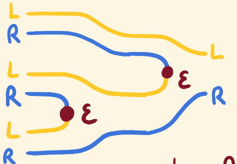
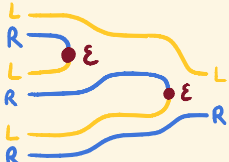
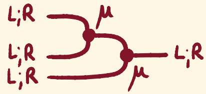
We want to define candidates multiplication and unit for $L;R$.

- *Candidate unit* :  := 
- *Candidate multiplication* :  := 

These are natural transformations because they are horizontal and vertical compositions of natural transformations.

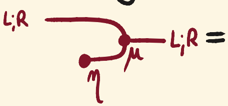
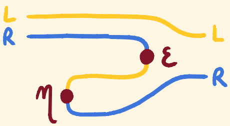
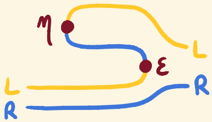
We need to show associativity and unitality.

- *Associativity.*

 =  =  = 

interchange law

- *Unitality.*

 =  =  =  = 

snake equations

$\square$

KLEISLI CATEGORY

KLEISLI CATEGORY

For a monad (T, μ, η) (or $(T, (-)^T, \eta)$) on a category $\mathcal{C}$, define its Kleisli category $\mathrm{Kl}(T)$ as follows.

- Objects are the objects of $\mathcal{C}$: $\mathrm{Kl}(T)_0 := \mathcal{C}_0$.
- Morphisms $f : A \nrightarrow B$ are morphisms $f : A \to TB$ in $\mathcal{C}$: $\mathrm{Kl}(T)(A,B) := \mathcal{C}(A, TB)$.
- Composition of $f : A \nrightarrow B$ and $g : B \nrightarrow C$ is $f \mathbin{⨾} g := f ; Tg ; \mu_C = f ; g^T$.
- Identities $\mathbb{1}_A := \eta_A$.

PROPOSITION

For a monad (T, μ, η) (or $(T, (-)^T, \eta)$) on a category $\mathcal{C}$, $\mathrm{Kl}(T)$ is a category.

PROOF

We use the monad in Kleisli form.

- Associativity.

$$
\begin{aligned}
f \mathbin{⨾} (g \mathbin{⨾} h) &:= f ; (g ; h^T)^T && \text{by def of } ⨾ \\
&= f ; g^T ; h^T && \text{by 3.} \\
&= (f \mathbin{⨾} g) \mathbin{⨾} h && \text{by def of } ⨾
\end{aligned}
$$

- Unitality.

$$
\begin{aligned}
f \mathbin{⨾} \mathbb{1}_B &:= f ; \eta_B^T && \text{by def of } ⨾ \text{ and } \mathbb{1} \\
&= f ; \mathbb{1}_{TB} && \text{by 1.} \\
&= f && \text{by identities}
\end{aligned}
\qquad
\begin{aligned}
\mathbb{1}_A \mathbin{⨾} f &:= \eta_A ; f^T && \text{by def of } ⨾ \text{ and } \mathbb{1} \\
&= f && \text{by 2.}
\end{aligned}
$$

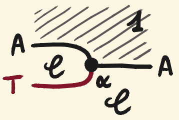
$\Rightarrow \mathrm{Kl}(T)$ is a category. $\square$

ALGEBRAS

ALGEBRA

An algebra for a monad (T, μ, η) on a category $\mathcal{C}$ is a pair (A, α) of

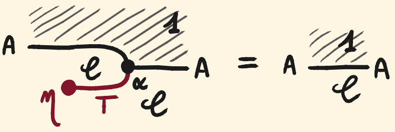
- an object $A \in \mathcal{C}_0$, and
- a morphism $\alpha : TA \to A$ in $\mathcal{C}$

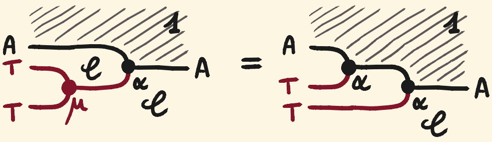


such that

$\eta_A ; \alpha = 1_A$ and $\mu_A ; \alpha = T\alpha ; \alpha$

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "1_A"'] & TA \arrow[d, "\alpha"] \\ & A \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} TTA \arrow[r, "\mu_A"] \arrow[d, "T\alpha"'] & TA \arrow[d, "\alpha"] \\ TA \arrow[r, "\alpha"'] & A \end{tikzcd}$$



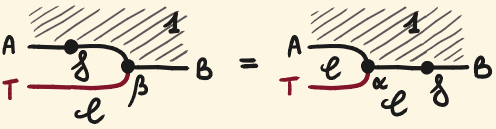


$\rightsquigarrow$ algebras for a monad (T, μ, η) are actions of the monoid (T, μ, η) on the 1-cell (functor) $!_A : 1 \to \mathcal{C}$, where 1 is the terminal category and $!_A(\cdot) := A$.

MORPHISM OF ALGEBRAS

For two algebras (A, α) and (B, β) for a monad (T, μ, η) on $\mathcal{C}$, a morphism $f : (A, \alpha) \to (B, \beta)$ is a morphism $f : A \to B$ in $\mathcal{C}$ such that $Tf ; \beta = \alpha ; f$

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "\alpha"'] & TB \arrow[d, "\beta"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$



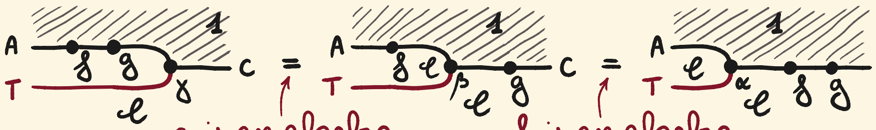
PROPOSITION

Algebras for a monad T and their morphisms form a category $EM(T)$, the Eilenberg-Moore category of the monad T.

PROOF

For $f : (A, \alpha) \to (B, \beta)$ and $g : (B, \beta) \to (C, \gamma)$, define $f ; g : (A, \alpha) \to (C, \gamma)$ as $f ; g$ in $\mathcal{C}$.

This defines an algebra morphism because

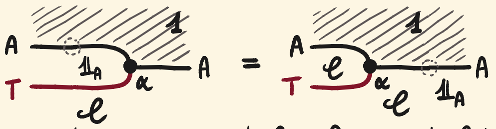


g is an algebra morphism

f is an algebra morphism

Composition is associative by associativity in $\mathcal{C}$: $f ; (g ; h) = (f ; g) ; h$.

For a T-algebra (A, α), the identity $1_{(A,\alpha)} : (A, \alpha) \to (A, \alpha)$ is given by the identity 1_A in $\mathcal{C}$, which is an algebra morphism because



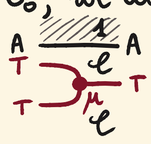
Composition is unital by unitality in $\mathcal{C}$: $f ; 1_{(A,\alpha)} = f = 1_{(A,\alpha)} ; f$.

$\square$

FREE ALGEBRAS

FREE ALGEBRA

For every object $A \in \mathcal{C}_0$, we can construct an algebra on TA:

the pair (TA, μ_A)  is an algebra because

$$\begin{tikzcd} TA \arrow[r, "\eta_{TA}"] \arrow[dr, "\mathbb{1}_{TA}"'] & TTA \arrow[d, "\mu_A"] \arrow[dl, phantom, "="] \\ & TA \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} TTTA \arrow[r, "\mu_{TA}"] \arrow[d, "T\mu_A"'] \arrow[dr, phantom, "="] & TTA \arrow[d, "\mu_A"] \\ TTA \arrow[r, "\mu_A"'] & TA \end{tikzcd}$$

by unitality and associativity of (T, μ, η).

$\rightsquigarrow$ free algebras for T form a subcategory $\mathrm{FAlg}(T)$ of $\mathrm{EM}(T)$.

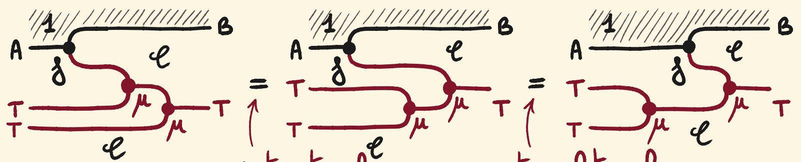
PROPOSITION

The category of free algebras $\mathrm{FAlg}(T)$ is equivalent to the Kleisli category $\mathrm{Kl}(T)$.

PROOF

Define a candidate equivalence $F : \mathrm{Kl}(T) \to \mathrm{FAlg}(T)$ as:
- for an object $A \in \mathrm{Kl}(T)_0 = \mathcal{C}_0$, $F(A) := (TA, \mu_A)$,
- for a morphism $f : A \to B$ in $\mathrm{Kl}(T)$, $F(f) := Tf ; \mu_B$ in $\mathcal{C}$.

We check that this gives a functor.
- F is well-defined because $F(A) := (TA, \mu_A)$ is a free algebra and $F(f) := Tf ; \mu_B$ is an algebra morphism:



associativity of μ naturality of μ

- F preserves composition:

$$\begin{aligned} F(f \fatsemi g) &:= T(f \fatsemi g) ; \mu_C && \text{by def of } F \\ &= T(f ; Tg ; \mu_C) ; \mu_C && \text{by def of composition in } \mathrm{Kl}(T) \\ &= Tf ; TTg ; T\mu_C ; \mu_C && \text{by functoriality of } T \\ &= Tf ; TTg ; \mu_{TC} ; \mu_C && \text{by associativity of } \mu \\ &= Tf ; \mu_B ; Tg ; \mu_C && \text{by naturality of } \mu \\ &=: F(f) ; F(g) && \text{by def of } F \end{aligned}$$

- F preserves identities:

$$\begin{aligned} F(\mathbb{1}_A) &:= T\eta_A ; \mu_A && \text{by def of } F \text{ and identities in } \mathrm{Kl}(T) \\ &= \mathbb{1}_{TA} && \text{by unitality of } \mu \end{aligned}$$

We show that F is an equivalence by finding a functor $G : \mathrm{FAlg}(T) \to \mathrm{Kl}(T)$ such that $F ; G \simeq \mathbb{1}_{\mathrm{Kl}T}$ and $G ; F \simeq \mathbb{1}_{\mathrm{FAlg}T}$.
For a free T-algebra (B, β), there is an object $U_B \in \mathcal{C}_0$ such that $B = TU_B$ and $\beta = \mu_{U_B}$. Assuming the axiom of choice, we can pick such a U_B and define $G(B, \beta) := U_B$.
For a morphism of free algebras $f : (B, \beta) \to (C, \gamma)$, define $G(f) := \eta_{U_B} ; f : U_B \to U_C$ in $\mathrm{Kl}(T)$.
This is well-defined and is a functor:
- for $f : (B, \beta) \to (C, \gamma)$ and $g : (C, \gamma) \to (D, \delta)$,

$$\begin{aligned} G(f) \fatsemi G(g) &:= (\eta_{U_B} ; f) \fatsemi (\eta_{U_C} ; g) && \text{by def of } G \\ &= \eta_{U_B} ; f ; T(\eta_{U_C} ; g) ; \mu_{U_D} && \text{by def of Kleisli composition} \\ &= \eta_{U_B} ; f ; T\eta_{U_C} ; Tg ; \mu_{U_D} && \text{by functoriality of } T \\ &= \eta_{U_B} ; f ; T\eta_{U_C} ; Tg ; \delta && \text{because } TU_D = D \text{ and } \delta = \mu_{U_D} \\ &= \eta_{U_B} ; f ; T\eta_{U_C} ; \gamma ; g && \text{because } g \text{ is an algebra morphism} \\ &= \eta_{U_B} ; f ; T\eta_{U_C} ; \mu_{U_C} ; g && \text{because } TU_C = C \text{ and } \gamma = \mu_{U_C} \\ &= \eta_{U_B} ; f ; \mathbb{1}_{TU_C} ; g && \text{by unitality of } \mu \\ &= \eta_{U_B} ; f ; g && \text{by identities} \\ &=: G(f ; g) && \text{by def of } G \end{aligned}$$

-
$$\begin{aligned} G(\mathbb{1}_{(B,\beta)}) &:= \eta_{U_B} ; \mathbb{1}_{TU_B} && \text{by def of } G \\ &= \eta_{U_B} && \text{by identities in } \mathcal{C} \\ &= \mathbb{1}_{G(B,\beta)} && \text{by def of identities in } \mathrm{Kl}(T) \end{aligned}$$

We define candidates natural isomorphisms $\alpha : G ; F \xrightarrow{\simeq} \mathbb{1}_{\mathrm{FAlg}T}$ and $\beta : F ; G \xrightarrow{\simeq} \mathbb{1}_{\mathrm{Kl}T}$ by
We have $FG(B, \beta) = F(U_B) = (TU_B, \mu_{U_B}) = (B, \beta)$.
Then, we can define $\alpha_{(B,\beta)} := \mathbb{1}_{(B,\beta)} : FG(B,\beta) \to (B,\beta)$.
This is a natural isomorphism.

We have $GF(A) = G(TA, \mu_A) = U_{TA}$, so we want $\beta_A : U_{TA} \to A$ in $\mathrm{Kl}(T)$, i.e. $\beta_A : U_{TA} \to TA$ in $\mathcal{C}$.
We have $\eta_{U_{TA}} : U_{TA} \to TU_{TA}$ in $\mathcal{C}$ and $TU_{TA} = TA$.
Then, we can define $\beta_A := \eta_{U_{TA}} = \eta_{G(TA, \mu_A)} = \eta_{GFA} = (\mathbb{1}_F \otimes \mathbb{1}_G \otimes \eta)_A$
This is a natural transformation because it is a composition of natural transformations.
It is also an isomorphism in $\mathrm{Kl}(T)$, with inverse η_A:

$$\begin{cases} \eta_A \fatsemi \eta_{U_{TA}} := \eta_A ; T\eta_{U_{TA}} ; \mu_{U_{TA}} = \eta_A ; \mathbb{1}_{TU_{TA}} = \eta_A ; \mathbb{1}_{TA} = \eta_A \\ \eta_{U_{TA}} \fatsemi \eta_A := \eta_{U_{TA}} ; T\eta_A ; \mu_A = \eta_{U_{TA}} ; \mathbb{1}_{TA} = \eta_{U_{TA}} ; \mathbb{1}_{TU_{TA}} = \eta_{U_{TA}} \end{cases}$$

Then, $\mathrm{Kl}(T)$ is equivalent to $\mathrm{FAlg}(T)$. $\square$

MORE ON KLEISLI EXTENSIONS

We have seen that conditions 1, 2 and 3 are sufficient for defining $Kl(T)$. They are also necessary.

PROPOSITION (hopefully answering Michele's question)

Suppose that we have
- a function $T : \mathcal{C}_0 \to \mathcal{C}_0$
- an operation $(-)^T : \mathcal{C}(A, TB) \to \mathcal{C}(TA, TB)$
- a family of morphisms $\eta_A : A \to TA$ in $\mathcal{C}$, for every $A \in \mathcal{C}_0$

such that morphisms $f : A \to TB$ form a category with composition $f \mathbin{⨾} g := f ; g^T$ and identities η_A.

TS: $(T, (-)^T, \eta)$ is a monad in Kleisli form.

PROOF

1. $\eta_A^T = 1_{TA} ; \eta_A^T$ by identities in $\mathcal{C}$
$= 1_{TA} \mathbin{⨾} \eta_A$ by def of ⨾
$= 1_{TA}$ because η_A identity for ⨾

2. $\eta_A ; f^T = \eta_A \mathbin{⨾} f$ by def of ⨾
$= f$ because η_A identity for ⨾

3. $f^T ; g^T = 1_{TA} ; f^T ; g^T$ by identities in $\mathcal{C}$
$= (1_{TA} \mathbin{⨾} f) ; g^T$ by def of ⨾
$= (1_{TA} \mathbin{⨾} f) \mathbin{⨾} g$ by def of ⨾
$= 1_{TA} \mathbin{⨾} (f \mathbin{⨾} g)$ by associativity of ⨾
$= 1_{TA} ; (f ; g^T)^T$ by def of ⨾
$= (f ; g^T)^T$ by def of ⨾

$\Rightarrow (T, (-)^T, \eta)$ is a monad in Kleisli form. $\square$