PLAN FOR TODAY

24/4/2024

PART 1

- recap : monads
- Kleisli categories
- distributive laws
- Kleisli are free algebras

PART 2

- monoidal monads
- strength & commutativity

PART 3

- structure in Par, Rel
- structure in Stoch, SubStoch

RECAP : MONADS

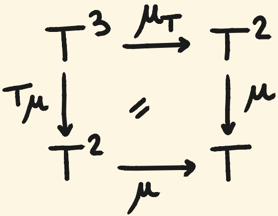
MONAD

(T, μ, η) is

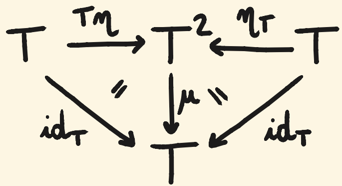
- $T : \mathcal{C} \to \mathcal{C}$ endofunctor
- $\mu : T \circ T \to T$ natural transformation (multiplication)
- $\eta : id_{\mathcal{C}} \to T$ natural transformation (unit)

such that

(associativity)



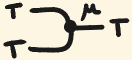
(unitality)



RECAP : MONADS

MONAD (= monoid in endofunctors)

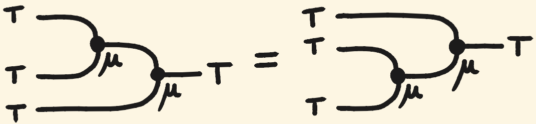
(T, μ, η) is

- $T \in (\mathcal{C}at(\mathcal{C}, \mathcal{C}), \circ, id_{\mathcal{C}})$
- 

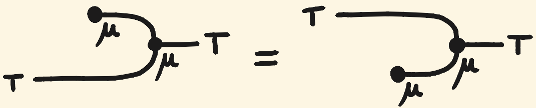
(multiplication)
- 

(unit)

such that



(associativity)



(unitality)

MONADS: EXAMPLES

Exception & maybe monads

$\mathscr{C}$ category with coproducts

$E \in \mathrm{obj}\,\mathscr{C}$

$$\begin{aligned} Ex : \mathscr{C} &\longrightarrow \mathscr{C} \\ A &\longmapsto A+E \end{aligned}$$ is a monad with

multiplication and unit given by injections:

$\mu_A : A+E+E \longrightarrow A+E$ and $\eta_A : A \longrightarrow A+E$

$\mu_A := id_A + [i_E, i_E]$ $\qquad$ $\eta_A := i_A$

$\rightsquigarrow$ $E = 1$ (terminal object) $\Rightarrow$ maybe monad

MONADS: EXAMPLES

Powerset monad

$$\mathcal{P} : \mathbf{Set} \longrightarrow \mathbf{Set} \qquad A \longmapsto \{X \subseteq A\}$$

$$\mu_A : \mathcal{P}^2 A \longrightarrow \mathcal{P}A \qquad X \longmapsto \bigcup_{Y \in X} Y$$

$$\eta_A : A \longrightarrow \mathcal{P}(A) \qquad a \longmapsto \{a\}$$

Distribution monad (finitary)

$$\mathcal{D} : \mathbf{Set} \longrightarrow \mathbf{Set} \qquad A \longmapsto \{\sigma : A \to [0,1] \mid \operatorname{supp} \sigma \text{ finite} \wedge \sum_a \sigma(a) = 1\}$$

$$\mu_A : \mathcal{D}^2 A \longrightarrow A \qquad \varphi \longmapsto \sum_\sigma \varphi(\sigma) \cdot \sigma(-)$$

$$\eta_A : A \longrightarrow \mathcal{D}A \qquad a \longmapsto \delta_a$$

↪ with ≤ 1 we get subdistributions

MONADS: EXAMPLES

State monad

$(\mathcal{C}, \times, 1, (-)^{(=)})$ cartesian closed category

$S \in \mathrm{obj}\,\mathcal{C}$ $\rightsquigarrow$ global state

$$St : \mathcal{C} \longrightarrow \mathcal{C}$$
$$A \longmapsto (S \times A)^S$$

$$\mu_A : (S \times (S \times A)^S)^S \longrightarrow (S \times A)^S$$
$$f : S \to S \times (S \to S \times A) \longmapsto \pi_2(f)(\pi_1(f)(-)) = \lambda s.\, f_1(f_2(s))$$
$$s \mapsto g(t) \text{ for } f(s) = (t, g)$$

$$\eta_A : A \longrightarrow (S \times A)^S$$
$$a \longmapsto \langle id_S, a \rangle = \lambda s.(s, a)$$

PROJECT IDEA: promonads for state on monoidal non-cartesian non-closed categories.

KLEISLI CATEGORIES: MOTIVATION

Monads add effects: TA = "T-effects on A"

ex $A+1$ adds failure to the set of resources

$\mathcal{D}(A)$ gives probability distributions over resources

We would like to consider computations $A \to T(B)$

= "computations with T-effects"

ex $f: A \to B+1$ are partial functions

$f: A \to \mathcal{D}(B)$ are stochastic maps

$\rightsquigarrow f(b|a) := f(a)(b)$ = "probability of b given a"

KLEISLI CATEGORIES

(T, μ, η) monad on $\mathcal{C}$

KLEISLI CATEGORY

$\mathrm{Kl}(T, \mu, \eta)$ category where (usually just $\mathrm{Kl}\,T$)

- objects are objects of $\mathcal{C}$
- morphisms $A \nrightarrow B$ are $A \to TB$ in $\mathcal{C}$
- composition $A \overset{f}{\nrightarrow} B \overset{g}{\nrightarrow} C := A \xrightarrow{f} TB \xrightarrow{Tg} T^2C \xrightarrow{\mu_C} TC$
- identities $A \overset{1_A}{\nrightarrow} A := A \xrightarrow{\eta_A} TA$

KLEISLI CATEGORIES

(T, μ, η) monad on $\mathscr{C}$

PROPOSITION

$\mathrm{Kl}\,T$ is a category.

PROOF

Exercise. Use associativity & unitality of (T, μ, η). $\square$

PROJECT IDEA: comonads & coKleisli categories

↳ <u>ex</u> streams on cartesian categories

INTERLUDE: MONADS IN KLEISLI FORM

MONAD IN KLEISLI FORM

$(T, (-)^T, \eta)$ is

- a function $T : \mathrm{obj}\,\mathcal{C} \to \mathrm{obj}\,\mathcal{C}$
- an operation $(-)^T : \mathcal{C}(A, TB) \to \mathcal{C}(TA, TB)$ — Kleisli extension
- a family $\eta_A : A \to TA$ for $A \in \mathrm{obj}\,\mathcal{C}$

such that

1. $(\eta_A)^T = id_{TA}$
2. $\eta_A ; f^T = f$
3. $f^T ; g^T = (f ; g^T)^T$

INTERLUDE: MONADS IN KLEISLI FORM

PROPOSITION

Monads are the same thing as monads in Kleisli form.

PROOF

Exercise. $\square$

KLEISLI CATEGORY

$\mathrm{Kl}(T, (-)^T, \eta)$ category where

- objects are objects of $\mathcal{C}$
- morphisms $A \nrightarrow B$ are $A \to TB$ in $\mathcal{C}$
- composition $A \overset{f}{\nrightarrow} B \overset{g}{\nrightarrow} C := A \xrightarrow{f} TB \xrightarrow{g^T} TC$
- identities $A \overset{1_A}{\nrightarrow} A := A \xrightarrow{\eta_A} TA$

EXAMPLES

- $\mathrm{Kl}(-+1) \simeq \mathrm{Par}$ category of sets and partial functions
- $\mathrm{Kl}(\mathcal{D}) \simeq \mathrm{Stoch}$ category of sets and stochastic maps

  ⇝ $\mathcal{G} : \mathrm{Meas} \to \mathrm{Meas}$ Giry monad and $\mathrm{Kl}\,\mathcal{G}$ for 'continuous' stochastic maps
- $\mathrm{Kl}(\mathcal{P}) \simeq \mathrm{Rel}$ category of sets and relations
- $\mathrm{Kl}(\mathrm{St})$ category of sets and stateful functions

  ⇝ $A \nrightarrow B \simeq S \times A \to S \times B$

COMPOSING MONADS

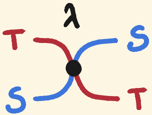
Subdistributions are distributions with failure:

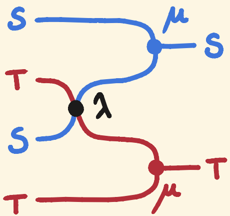
$$\mathcal{D}(A+1) \simeq \mathcal{D}_{\leq 1}(A) := \{\sigma : A \to [0,1] \mid \operatorname{supp} \sigma \text{ finite} \wedge \sum_a \sigma(a) \leq 1\}$$

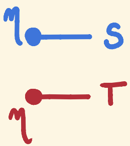
$(\mathcal{C}at(\mathcal{C},\mathcal{C}), \circ, id_{\mathcal{C}})$ is not symmetric

⇒ we cannot compose monads in general

⇒ we need a well-behaved 'swap' in $\mathcal{C}at(\mathcal{C},\mathcal{C})$







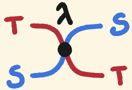
DISTRIBUTIVE LAWS

$(\mathcal{C}, \otimes, I)$ monoidal category

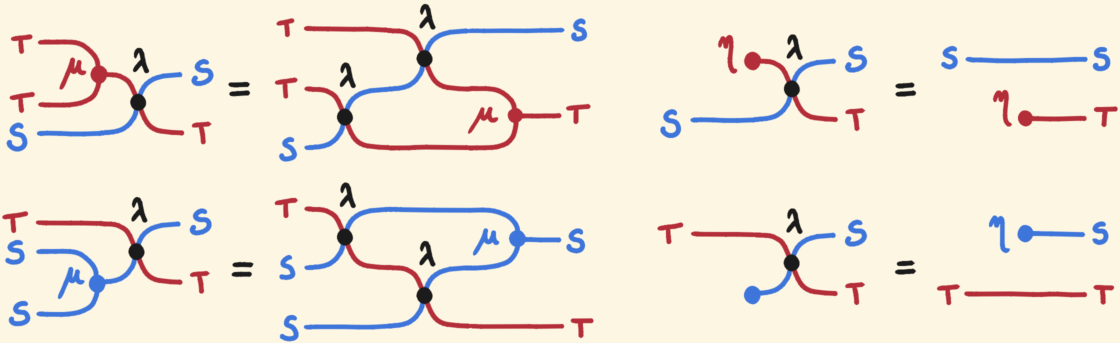
$(T,$  ,  $)$ and $(S,$  ,  $)$ monoids

DISTRIBUTIVE LAW OF MONOIDS

$\lambda : T \otimes S \to S \otimes T$



such that

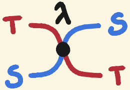


DISTRIBUTIVE LAWS

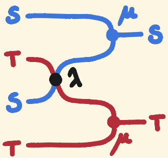
$(\mathcal{C}, \otimes, I)$ monoidal category

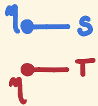
$(T$, , $)$ and $(S$, , $)$ monoids

PROPOSITION



distributive law

$\Rightarrow$ $(S \otimes T$, 

, 

$)$ is a monoid

PROOF

Exercise. □

PROJECT IDEA: Weak distributive laws

DISTRIBUTIVE LAWS: EXAMPLES

- $\lambda_A : \mathcal{D}(A) + 1 \longrightarrow \mathcal{D}(A+1)$ ⇝ subdistributions

  $\lambda_A(\sigma) := \sigma \qquad \lambda_A(\bot) := \delta_\bot$

- $\lambda_A : \mathcal{P}_{NE}(A+1) \longrightarrow \mathcal{P}_{NE}(A) + 1 \simeq \mathcal{P}$ ⇝ relations

  $\lambda_A(X) := X \setminus \{\bot\} \qquad \lambda_A(\{\bot\}) := \bot$

- $\lambda_A : \mathcal{P}_{NE}(A) + 1 \longrightarrow \mathcal{P}_{NE}(A+1)$ ⇝ 'relations' with explicit failure

  $\lambda_A(X) := X \qquad \lambda_A(\bot) := \{\bot\}$

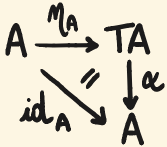
PROJECT IDEA: Zwart & Marsden (2022)
No-go theorems for distributive laws

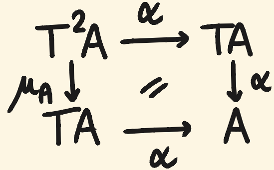
RECALL: FREE ALGEBRAS

(T, μ, η) monad on $\mathcal{C}$

ALGEBRA FOR A MONAD

$(A, \alpha : TA \to A)$ such that





FREE ALGEBRA

$(TA, \mu_A : T^2A \to TA)$

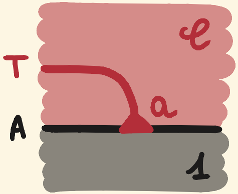
PROPOSITION

Free algebras are algebras.

PROOF

Exercise. $\square$

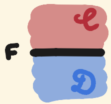
INTERLUDE: COLOURING REGIONS



specify morphisms in <u>bicategories</u> (PROJECT IDEA)

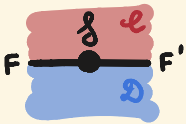
0-cells

1-cells



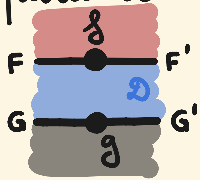
$F : \mathcal{C} \to \mathcal{D}$

2-cells



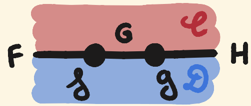
$f : F \Rightarrow F'$

parallel



$f \otimes g : F \otimes G \to F' \otimes G'$

sequential



$f ; g : F \to H$

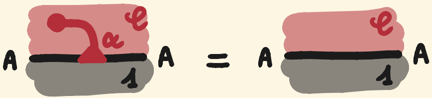
<u>ex</u> Cat with categories, functors, natural transformations

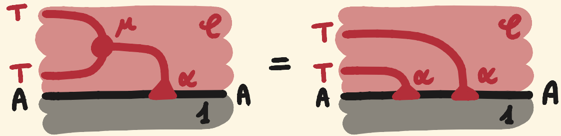
ALGEBRAS ARE MODULES

(T, , ) monad on $\mathcal{C}$

ALGEBRA FOR A MONAD

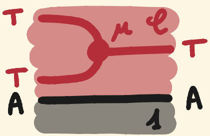
(A, ) such that





⇝ objects of $\mathcal{C}$ are functors $A : 1 \to \mathcal{C}$
arrows $A \to B$ are natural transformations $f : A \Rightarrow B$

FREE ALGEBRA

(TA, )

FREE ALGEBRAS ARE KLEISLI

$T : \mathcal{C} \to \mathcal{C}$ monad

CATEGORIES OF ALGEBRAS

- Alg_T of algebras and their morphisms
- $\mathrm{FAlg}_T \subseteq \mathrm{Alg}_T$ of free algebras and their morphisms

  ↑ full

THEOREM

There's an equivalence of categories

$$\mathrm{FAlg}_T \simeq \mathrm{Kl}(T)$$

FREE ALGEBRAS ARE KLEISLI

PROOF SKETCH

Define the candidate equivalence $F : Kl(T) \to FAlg_T$:

- on objects $F(A) := (TA, \mu_A)$
- on morphisms $f : A \nrightarrow B \equiv f : A \to TB$
  as $F(f) := Tf ; \mu_B = f^T : TA \to TB$

We need to show:

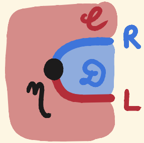
1. F is well-defined (i.e. $F(f)$ is an algebra morph)
2. F is a functor
3. F is an equivalence (e.g. F is full, faithful and essentially surjective on objects)

Details as exercise. □

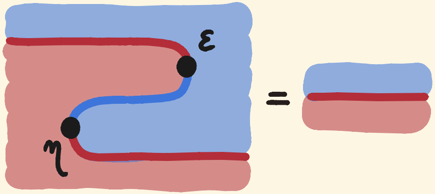
RECALL: ADJUNCTIONS & MONADS

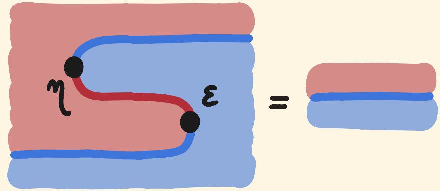
ADJUNCTION

$\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$ ($\perp$) is 

and 

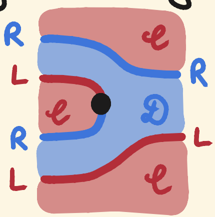
such that

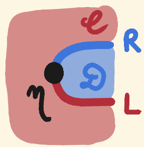


and


THEOREM (PAIR OF PANTS MONADS)

Every adjunction gives a monad.

($R \circ L$, 

, 

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RECALL: ADJUNCTIONS & MONADS

$T : \mathcal{C} \to \mathcal{C}$ monad

THEOREM

Every monad arises from an adjunction.

$$\mathcal{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathrm{Alg}_T \qquad (F \dashv U)$$

$F(A) := (TA, \mu_A)$ $\qquad$ $F(f) := T(f)$

$U(A, \alpha) := A$ $\qquad$ $U(f) := f$

KLEISLI ADJUNCTION

$T : \mathcal{C} \to \mathcal{C}$ monad

THEOREM

Every monad arises from an adjunction.

$$\mathcal{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathrm{Kl}(T) \quad (\simeq \mathrm{FAlg}_T \subseteq \mathrm{Alg}_T)$$

with $F \dashv U$.

PROOF SKETCH

$F(A) := A$ $\qquad$ $F(f) := f ; \eta_B$

$U(A) := TA$ $\qquad$ $U(f) := Tf ; \mu_B = f^T$

Details as exercise. $\square$

ADJUNCTIONS FOR A MONAD

PROJECT IDEAS: Beck's monadicity theorem
category of adjunction-resolutions of a monad

PART 1

- recap : monads
- Kleisli categories
- distributive laws
- Kleisli are free algebras

PART 2

- monoidal monads
- strength & commutativity

PART 3

- structure in Par, Rel
- structure in Stoch, SubStoch

MONOIDAL KLEISLI CATEGORIES

$T : \mathcal{C} \to \mathcal{C}$ monad

$\mathrm{Kl}\,T$ $\rightsquigarrow$ category of computations with T-effects

Q: can we compose these computations in parallel?

= is $\mathrm{Kl}\,T$ monoidal?

$$f \boxtimes f' := A \otimes A' \xrightarrow{f \otimes f'} TB \otimes TB' \xrightarrow{?} T(B \otimes B')$$

A: YES, if

$$\begin{cases} (\mathcal{C}, \otimes, I) & \text{monoidal category} \\ (T, \mu, \eta) & \text{monoidal monad (a.k.a. commutative monad)} \end{cases}$$

MONOIDAL MONADS

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ monoidal category

MONOIDAL MONAD

Monoid in $\mathrm{MonCat}(\mathcal{C}, \mathcal{C})$ (with lax monoidal functors).

MONOIDAL MONADS

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ monoidal category

MONOIDAL MONAD

Monoid in $\mathrm{MonCat}(\mathcal{C}, \mathcal{C})$:

- (T, m, e) lax monoidal endofunctor on $(\mathcal{C}, \otimes, I)$
- $\begin{cases} \mu : T^2 \Rightarrow T \\ \eta : id_{\mathcal{C}} \Rightarrow T \end{cases}$ monoidal natural transformations

  satisfying associativity and unitality.

MONOIDAL MONADS

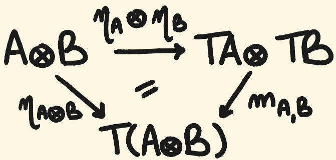
$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ monoidal category

MONOIDAL MONAD (USEFUL DEFINITION)

(T, μ, η, m) is

- (T, μ, η) monad on $\mathcal{C}$
- $m : T(-) \otimes T(=) \Rightarrow T(- \otimes =)$ natural transformation

satisfying



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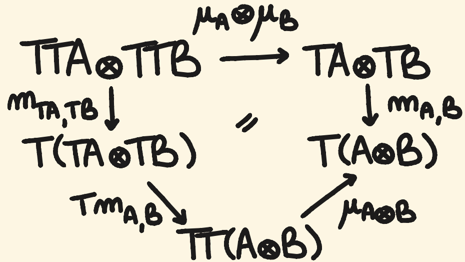


MONOIDAL KLEISLI CATEGORIES

THEOREM

The Kleisli category of a monoidal monad is monoidal.

PROOF SKETCH

T monoidal monad on $(\mathcal{C}, \otimes, I)$.

Define a candidate monoidal structure on $\mathcal{K}\ell T$:

- monoidal unit is I
- monoidal product on objects is $A \boxtimes B := A \otimes B$
- monoidal product on arrows is

$$f \boxtimes f' := A \otimes A' \xrightarrow{f \otimes f'} TB \otimes TB' \xrightarrow{m_{B,B'}} T(B \otimes B')$$

Details as exercise. □

EXAMPLES

- Exceptions & maybe

  $(E, \cdot, u)$ monoid $\rightsquigarrow$ $(1, !, !)$ is always a monoid

  $$m_{A,B} : (A+E)\times(B+E) \longrightarrow A\times B + A\times E + E\times B + E\times E \longrightarrow A\times B + E$$

- Powerset

  $$m_{A,B} : \mathcal{P}A \times \mathcal{P}B \longrightarrow \mathcal{P}(A\times B)$$
  $$(X, Y) \longmapsto X\times Y$$

- Distributions

  $$m_{A,B} : \mathcal{D}A \times \mathcal{D}B \longrightarrow \mathcal{D}(A\times B)$$
  $$(\sigma, \tau) \longmapsto \sigma\cdot\tau$$

EXAMPLES

- NON-EXAMPLE: state monads

$$(S\times A)^S \times (S\times B)^S \xrightarrow{?} (S\times A\times B)^S$$ (details later)

- every monad is monoidal wrt coproducts

PROPOSITION

$\begin{cases} (T,\mu,\eta) \text{ monad on } \mathscr{C} \\ \mathscr{C} \text{ has coproducts} \end{cases}$

$\Rightarrow$ T is monoidal wrt $(\mathscr{C}, +, 0)$

PROOF SKETCH

$m_{A,B} : TA + TB \xrightarrow{[Ti_A,\, Ti_B]} T(A+B)$. Details as exercise. □

STRONG & COMMUTATIVE MONADS

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ monoidal category

STRONG MONAD

(T, μ, η, s^L, s^R) is

- (T, μ, η) monad on $\mathcal{C}$
- $\begin{cases} s^L : (-) \otimes T(=) \Rightarrow T(- \otimes =) \\ s^R : T(-) \otimes (=) \Rightarrow T(- \otimes =) \end{cases}$ natural transformations

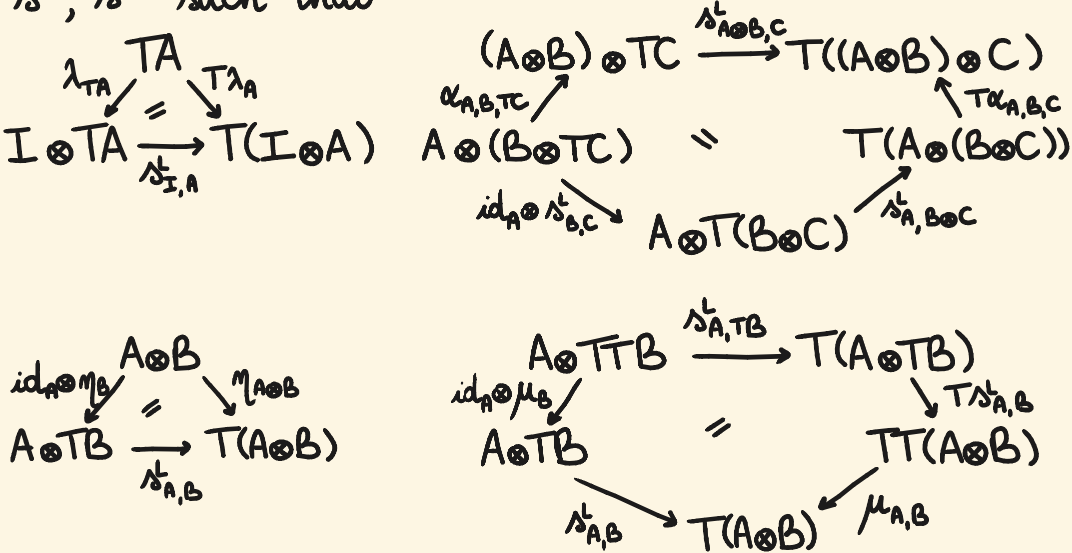
that play well with unitors, associator, monad multiplication and unit.

When $\mathcal{C}$ is symmetric, s^L is enough:

$$s^R_{A,B} := TA \otimes B \xrightarrow{\sigma} B \otimes TA \xrightarrow{s^L} T(B \otimes A) \xrightarrow{T\sigma} T(A \otimes B)$$

STRONG & COMMUTATIVE MONADS

s^L, s^R such that



+ similar ones for s^R

STRONG & COMMUTATIVE MONADS

PROPOSITION

Every monad on $(Set, \times, 1)$ is strong.

$$s^L_{A,B} := A \times TB \xrightarrow{u \times id} (B \to A \times B) \times TB \xrightarrow{ev} T(A \times B)$$

$$(a, b) \longmapsto (\langle a, id_B \rangle, b) \longmapsto T(\langle a, id_B \rangle)(b)$$

THEOREM

For a strong monad T, $Kl(T)$ is premonoidal.

$$TA \otimes TB \xrightarrow{s^L_{TA,B}} T(TA \otimes B) \xrightarrow{Ts^R_{A,B}} TT(A \otimes B) \xrightarrow{\mu_{A,B}} T(A \otimes B)$$

$$\neq$$

$$TA \otimes TB \xrightarrow{s^R_{A,B}} T(A \otimes TB) \xrightarrow{Ts^L_{A,B}} TT(A \otimes B) \xrightarrow{\mu_{A,B}} T(A \otimes B)$$

STRONG & COMMUTATIVE MONADS

COMMUTATIVE MONAD

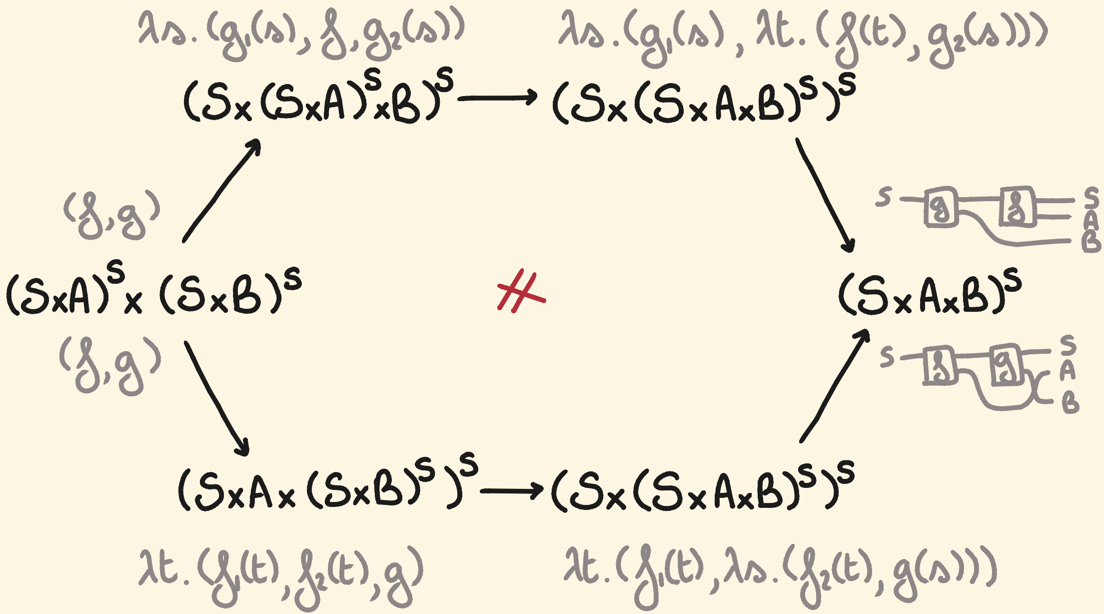
(T, μ, η, s^L, s^R) strong monad such that

$$
\begin{array}{ccccccc}
 & & T(TA \otimes B) & \xrightarrow{T s^R_{A,B}} & TT(A \otimes B) & & \\
 & \overset{s^L_{TA,B}}{\nearrow} & & & & \overset{\mu_{A,B}}{\searrow} & \\
TA \otimes TB & & & /\!/ & & & T(A \otimes B) \\
 & \underset{s^R_{A,B}}{\searrow} & & & & \underset{\mu_{A,B}}{\nearrow} & \\
 & & T(A \otimes TB) & \xrightarrow{T s^L_{A,B}} & TT(A \otimes B) & &
\end{array}
$$

THEOREM

Commutative monads are the same thing as monoidal monads.

STATE MONADS : A NON-EXAMPLE



PART 1

- recap : monads
- Kleisli categories
- distributive laws
- Kleisli are free algebras

PART 2

- monoidal monads
- strength & commutativity

PART 3

- structure in Par, Rel
- structure in Stoch, SubStoch

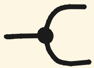
RECALL: FOX & STRUCTURE IN SET

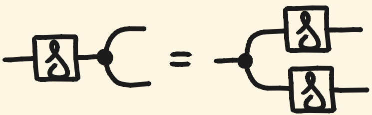
FOX'S THEOREM

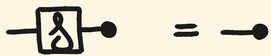
$(\mathcal{C}, \otimes, I)$ cartesian $\Leftrightarrow$ every object has a coherent natural comonoid structure.

- $(\mathcal{S}et, \times, 1)$ is cartesian

  $\Rightarrow$ we have copy and discard

 and  such that



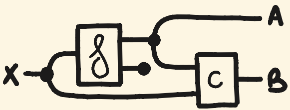


STRUCTURE IN KLEISLI CATEGORIES

(T, μ, η) monad on $\mathcal{S}et$

$i : \mathcal{S}et \hookrightarrow \mathcal{K}\ell T$

$\Rightarrow \mathcal{K}\ell T$ is a copy-discard category

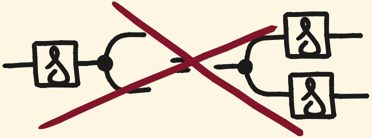


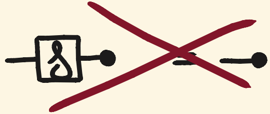

```
cond(x) = do
  f(x) → (a, b)
  c(a, x) → b'
  return (a, b')
```

⇝ variables can be copied and discarded

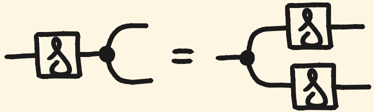
DETERMINISTIC & TOTAL ARROWS

In general, in KlT



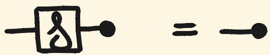


Copiable arrows are deterministic



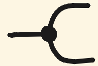
Discardable arrows are total

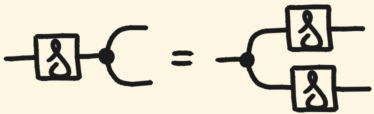
(causal in quantum)

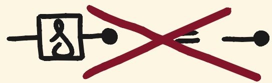


PARTIAL FUNCTIONS

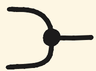
$\mathrm{Par} \simeq \mathrm{Kl}(-+1)$

 and  such that





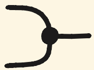
There's also the equality check

 $: A \times A \longrightarrow A+1$

$$(a,a') \longmapsto \begin{cases} a & \text{if } a = a' \\ \bot & \text{if } a \neq a' \end{cases}$$

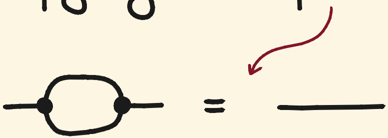
AXIOMS FOR PARTIAL FUNCTIONS

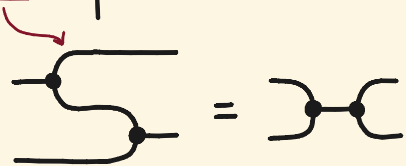
cocommutative comonoid



associative

satisfying the special Frobenius equations





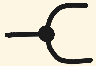
PROJECT IDEA: discrete cartesian restriction categories

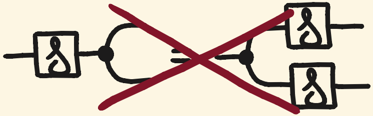
Cockett & Lack (2002-2007)

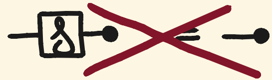
Restriction categories I-III

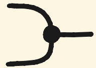
RELATIONS

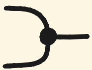
$\mathrm{Rel} \simeq \mathrm{Kl}\,P$

 and  such that





There's also a monoid ( , )

 $: A \times A \longrightarrow P(A)$

$$(a,a') \longmapsto \begin{cases} \{a\} & \text{if } a = a' \\ \emptyset & \text{if } a \neq a' \end{cases}$$

 $: 1 \longrightarrow P(A)$

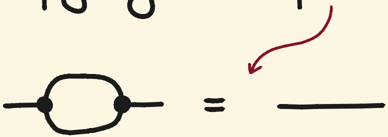
$$* \longmapsto A$$

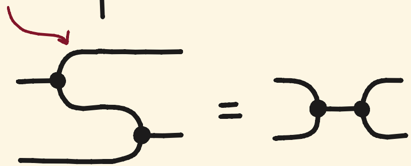
AXIOMS FOR RELATIONS

 cocommutative comonoid

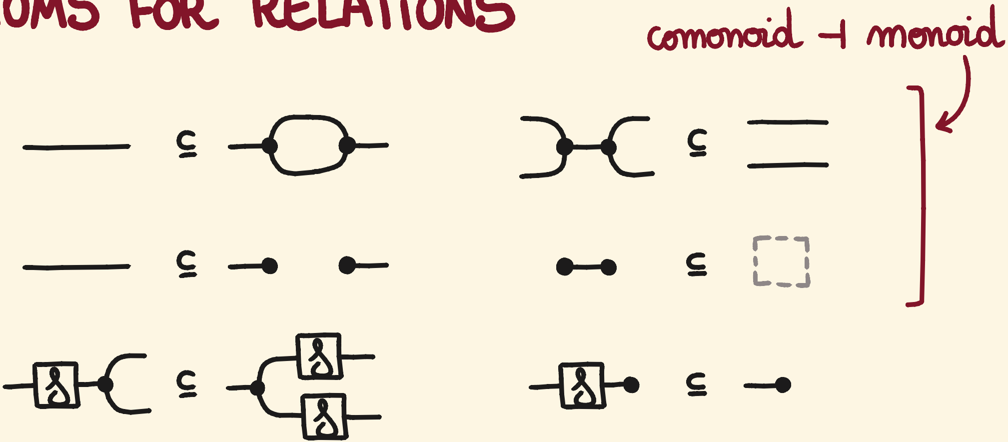
 commutative monoid

satisfying the special Frobenius equations





AXIOMS FOR RELATIONS

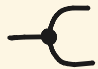


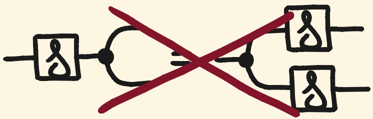
PROJECT IDEAS: Cartesian bicategories of relations

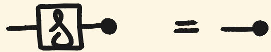
- Carboni & Walters (1987)
- a lot of work by Bonchi & coauthors

STOCHASTIC FUNCTIONS

$\mathsf{Stoch} \simeq \mathsf{Kl}\mathcal{D}$

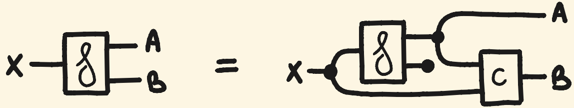
 and  such that





There are conditionals:

$\forall f : X \to A \otimes B \;\; \exists\, c : A \otimes X \to B$ st

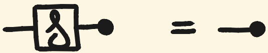


$$f(a, b \mid x) = c(b \mid a, x) \cdot \sum_{b' \in B} f(a, b' \mid x)$$

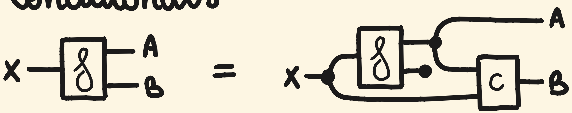
AXIOMS FOR STOCHASTIC FUNCTIONS

cocommutative comonoid

arrows are total



conditionals



PROJECT IDEAS: Markov categories   Fritz (2020)
Quasi-Borel spaces & lazyPPL
Heunen, Kammar, Staton, Yang (2017)

PARTIAL STOCHASTIC FUNCTIONS

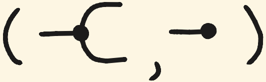
$\mathrm{SubStoch} \simeq \mathrm{Kl}\,\mathcal{D}(-+1)$

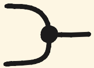
$f : X \to\!\!\!\!\!\!\!+\; A$ gives

$$\begin{cases} f(a|x) = \text{"probability of } a \text{ given } x\text{"} \\ f(\bot|x) = \text{"probability of failure given } x\text{"} \end{cases}$$

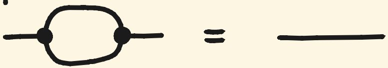
$\rightsquigarrow$ inherit structure from Par and Stoch

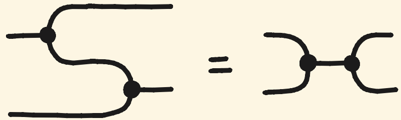
AXIOMS FOR PARTIAL STOCHASTIC FUNCTIONS

 cocommutative comonoid

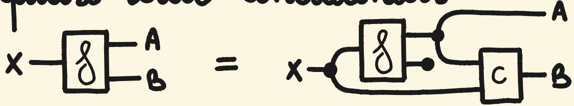
 associative

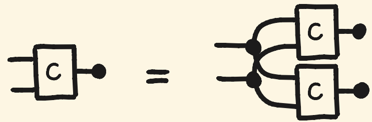
special Frobenius





quasi-total conditionals



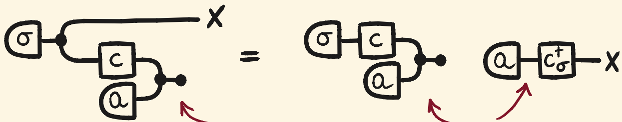


PROJECT IDEA: Partial Markov categories

EDL & Román (2023)

SYNTHETIC BAYES THEOREM

A deterministic observation $a : I \to A$ from a prior $\sigma : I \to X$ through a channel $c : X \to A$ determines an update proportional to the Bayes inversion $c^\dagger_\sigma$ evaluated on a.

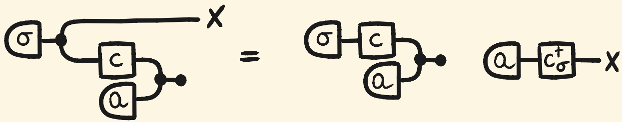


$$P(X{=}x \mid A{=}a) = \frac{P(A{=}a \mid X{=}x) \cdot P(X{=}x)}{\sum_{y \in X} P(A{=}a \mid X{=}y) \cdot P(X{=}y)}$$

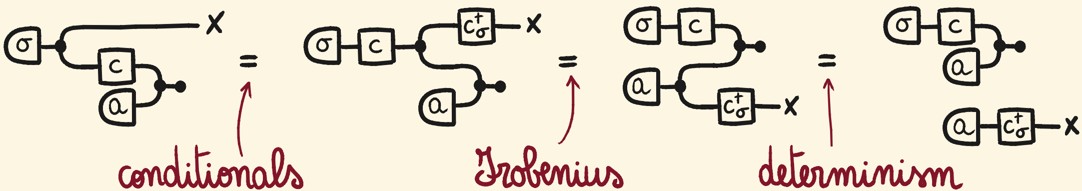
classical formula for Bayes theorem

SYNTHETIC BAYES THEOREM

A deterministic observation $a : I \to A$ from a prior $\sigma : I \to X$ through a channel $c : X \to A$ determines an update proportional to the Bayes inversion $c^{\dagger}_{\sigma}$ evaluated on a.



PROOF

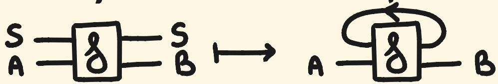


□

BONUS

FEEDBACK LOOPS

A feedback monoidal category is
a monoidal category $(\mathcal{C}, \otimes, I)$
with an operation $\mathrm{FBK} : \mathcal{C}(S \otimes A, S \otimes B) \longrightarrow \mathcal{C}(A, B)$



+ axioms

PROJECT IDEAS: feedback, Mealy machines & trace semantics

- Katis, Sabadini, Walters (1997, 2000, 2002)
- Bonchi, Piedeleu, Sobociński, Zanasi (2014, 2015, 2017)
- EDL, de Felice, Román (2022)

FEEDBACK LOOPS

FBK such that